THE

LONDON, EDINBURGH, AND DUBLIN

PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE.

[SEVENTH SERIES.]

NOVEMBER 1932.

LXXIX. *The Reduction of Platinum Resistance Thermometers to the International Temperature Scale. By* G. S. CALLENDAR *.

IN cases where an accurate temperature is required, the platinum resistance thermometer is in general use; it is especially useful for calibrating other forms of thermometer, such as thermocouples.

The following notes describe methods used for the reduction of the observations of resistance change to gas scale temperature, and are intended to save time in those reductions, and eliminate the use of log tables. Some remarks are also made about the accuracy of individual measurements.

(a) *International Temperature Scale.*

Temperature on the international scale from 0 to 660° C. is defined by the expression

$$R_t = R_0(1 + At + Bt^2), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where R_t is the resistance of a standard platinum thermometer at $t°$ C., R_0 is the resistance at 0° C., A and B are constants determined at the ice, steam, and sulphur points.

To evaluate $t°$ from (1) is obviously very troublesome, as the solution,

$$t = \frac{-A + \sqrt{A^2 - 4B\left(1 - \frac{R_t}{R_0}\right)}}{2B}, \quad . \quad . \quad . \quad (1')$$

* Communicated by A. C. Egerton, F.R.S.

involves finding the above root with some degree of accuracy.

It was long ago suggested * that, for simplicity of reduction, it was better first to obtain the platinum scale temperature "*pt*."

(b) *Platinum Scale Temperature.*

This scale is defined by the expression

$$pt = \frac{100(R_t - R_0)}{R_{100} - R_0}, \quad \ldots\ldots \quad (2)$$

where R_t, R_{100}, R_0 are the observed resistances of the thermometer at t, 100, 0° C., and $R_{100} - R_0$ is termed the "fundamental interval."

It is usual to arrange the thermometer so that the fundamental interval, expressed in units of the bridge used for measuring its resistance, is very near to some convenient multiple of 100. Thus, if the bridge unit was 1/20 ohm, the thermometer might be wound so that its fundamental interval was 5 ohm. But in practice it will, nearly always, be found that the fundamental interval is not exactly equal to 100 units of resistance on the bridge.

Let $R_{100} - R_0 = 100 - r$ bridge units. Then, from (2),

$$pt = \frac{100(R_t - R_0)}{100 - r} = (R_t - R_0)\left(1 + \frac{r}{100 - r}\right).$$

The quantity

$$\frac{r}{100 - r}$$

may be called the fundamental interval correction.

Let

$$\frac{r}{100 - r} = I.$$

Then

$$pt = (R_t - R_0)(1 + I), \quad \ldots\ldots \quad (3)$$

and *pt* can be obtained with an ordinary slide-rule, provided r has been adjusted to less than $\frac{1}{2}$ per cent. of $(R_{100} - R_0)$.

It is unnecessary to find the exact value of r, as "I" may be found directly from the resistance at the ice and steam points, as follows :—

* H. L. Callendar, Phil. Trans. 1887.

If t_s is the temperature of the steam, then

$$pt_s = t_s - 0{\cdot}015(t_s - 100),$$

as t_s is nearly 100° (see (7)), and

$$I = \frac{pt_s - (R_{t_s} - R_0)}{R_{t_s} - R_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

(c) *Conversion to "Gas" Scale.*

In order to find the difference between the two scales at any value of $t°$, substitute from (1) in expression (2),

$$pt = \frac{At + Bt^2}{A + 100B}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and the difference between the platinum scale and the gas scale from the above expression becomes

$$t - pt = \frac{B}{A + 100B}(100 - t)t. \quad . \quad . \quad . \quad . \quad (6)$$

This is called the "difference formula," and the constant

$$\frac{B}{A + 100B}$$

is usually written δ or D.

As pt is less than t, when t is greater than 100, B has a negative value, and the difference formula is written

$$t = pt + D(t - 100)t. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This expression (7) is a quadratic in t, the solution of which is

$$t = \frac{(1 + 100D) - \sqrt{(1 + 100D)^2 - 4Dpt}}{2D} \quad . \quad . \quad (8)$$

No advantage appears to have been obtained from the original expression (1′) for t. However, the constant D in (7) varies only slightly for different samples of platinum wire, and it is customary to use tables giving the value of $t - pt$ for each degree pt, for some normal value of D, such as 0·00015. With the aid of this table a temperature is at once obtainable, from any value of pt :

$$t = pt + 0{\cdot}00015(t - 100)t. \quad . \quad . \quad . \quad . \quad (9)*$$

* In some cases it may be necessary to assume that the value of D is the standard value of 0·00015, in which case the relatively large error of 0·3° at 400° may arise. See expression (11).

The true value of D is defined from the value of pt obtained at the sulphur boiling-point. That is, under a pressure of one normal atmosphere, when $t_s=444{\cdot}60$,

$$\mathrm{D}=\frac{444{\cdot}600-pt_s}{153209}. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Unfortunately, the value so obtained will seldom be sufficiently near to 0·00015 to enable the corresponding value of $t-pt$ to be obtained from the above-mentioned table, with an accuracy of $\pm 1/100°$ C.

It is therefore necessary to find the simplest possible method of evaluating the error in t, arising from the assumption that $\mathrm{D}=0{\cdot}00015$. Differentiating equation (7) with regard to D, the resulting value of $dt/d\mathrm{D}$ will give the required error in t, when multiplied by the appropriate defect of D from 0·00015. From (7),

$$\frac{dt}{d\mathrm{D}}=\frac{(t-100)t}{1-\mathrm{D}(2t-100)}. \quad . \quad . \quad . \quad . \quad (11)$$

A table of $dt/d\mathrm{D}$ can be made out for a number of different temperatures, taking the normal value of $\mathrm{D}=0{\cdot}00015$. Let this function be F.

To use this table in practice, the value of t, say t', from the table of $t-pt$ is obtained:

$$t'=pt+0{\cdot}00015(t'-100)t';$$

F at t' is found from a table of F and

$$t=t'+(\mathrm{D}-0{\cdot}00015)\mathrm{F}$$

to $\pm 1/500°$ at 600° for all values of D near the normal value. If great accuracy is required at temperatures of 600° and over, F can, of course, be taken from the table at $t°$ by a second approximation, but this is hardly ever necessary. The effect on the value of F of the small differences of D, from the mean value can be evaluated as follows from (11):

$$\frac{d\mathrm{F}}{d\mathrm{D}}=\frac{(2t-100)(t-100)t}{[1-\mathrm{D}(2t-100)]^2} \text{ to first approximation,} \quad (12)$$

and if $\Delta' t$ is the change in t, resulting from the small error $d\mathrm{F}$ in F,

$$\Delta' t=(\mathrm{D}-0{\cdot}00015)^2\frac{d\mathrm{F}}{d\mathrm{D}}, \text{ say } 10^{-12}\frac{d\mathrm{F}}{d\mathrm{D}}, \quad . \quad (13)$$

and it will be seen that one table of F will be sufficient for all normal thermometers, except at the highest temperatures.

With regard to the size of the table of F, it will be shown in the following that it will be rather extensive. Let Δt be the correction to t required by the defect of D from the normal,

$$\Delta t = (\mathrm{D} - 0{\cdot}00015)\mathrm{F}. \quad . \quad . \quad . \quad . \quad (14)$$

Also from (11),

$$\frac{d\mathrm{F}}{dt} = \frac{(2t-100) + 2\mathrm{D}\left(\frac{dt}{d\mathrm{D}}\right)}{1-\mathrm{D}(2t-100)}, \quad . \quad . \quad . \quad (15)$$

from which, if $t=600°$, $d\mathrm{F}/dt=1450$, and if $\mathrm{D}-0{\cdot}00015 = 10^{-6}$, then $\Delta'' t = 0{\cdot}0015°$, and F will have to be available for each degree, at this part of the scale.

In practice it will be found that Δt may amount to more than 0·5° C. at the higher temperature if D differs much from normal. To avoid the rather extensive table necessary for F or $dt/d\mathrm{D}$ it is generally better to use other methods for correction to true scale, for it is already necessary to look in tables for the value of $(t-pt)_{1\cdot5}$. The principles on which these methods can be based will now be considered.

(i.) *The Correction as a Coefficient of* $(t-pt)_{1\cdot5}$.

In many text-books the correction to t, resulting from the fact that D does not quite equal 0·00015, is given as a constant fraction of $0{\cdot}00015(t'-100)t'$ as taken from the $t-pt$ table.

Let $\mathrm{X} = 0{\cdot}00015(t'-100)t'$ or $t-pt$ from the table (when $\mathrm{D}=0{\cdot}00015$), then it is assumed

$$t-pt = \frac{\mathrm{D}}{0{\cdot}00015}\mathrm{X}$$

or

$$t = pt + \mathrm{X}(1+\Delta\mathrm{D}x), \quad . \quad . \quad . \quad . \quad (16)$$

where $\Delta\mathrm{D} = \mathrm{D}-0{\cdot}00015$. The value of x given is $1/0{\cdot}00015$ * ; this value was first suggested by Heycock and Neville some 35 years ago. From the above expression

$$dt/d\mathrm{D} = \mathrm{X}x, \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

* It may be noted that if dt is the difference from true scale, caused by taking x as a constant, equal to $1/0{\cdot}00015$,

$$dt = (x - 1/0{\cdot}00015)\Delta\mathrm{D}(t-pt),$$

and if $\Delta\mathrm{D} = 2\times10^{-6}$ and $t=400°$, dt will be $\pm{\cdot}03$.

If a table of this correction is given, on the basis of x equals the above constant, the true value of the correction may be obtained by multiplying the value from the table by dt/dpt.

and from (7) or (11)

$$dt/d\mathrm{D} = \frac{(t-100)t}{1-\mathrm{D}(2t-100)},$$

so that

$$x = \frac{(t-100)t}{\mathrm{X}}\left(\frac{1}{1-\mathrm{D}(2t-100)}\right). \quad . \quad . \quad (18)$$

By definition

$$\mathrm{D}(t-100)t = \mathrm{X}(1+\Delta\mathrm{D}x). \quad . \quad . \quad . \quad (19)$$

Substituting for $(t-100)t$ from (19) in (18)

$$x = \frac{1+\Delta\mathrm{D}x}{\mathrm{D}-\mathrm{D}^2(2t-100)},$$

and simplifying

$$x = \frac{1}{0{\cdot}00015-\mathrm{D}^2(2t-100)}, \quad . \quad . \quad . \quad (20)$$

and so x cannot be taken as a constant.

As $x=\mathrm{F}/\mathrm{X}$, it will, however, vary little with t, as compared with F.

The effect on the value of the function x, of the normal small differences of D from the mean value, will, as in the case of F, be small (see (12)) ; the resulting effect on t will generally be negligible as the second power of $\Delta\mathrm{D}$ enters here. It can be shown that

$$\frac{dx}{d\mathrm{D}} = \frac{(2t-100)}{0{\cdot}00015}\left(\frac{dt}{dpt}\right)^2; \quad . \quad . \quad . \quad . \quad (21)$$

also

$$\frac{dt}{dx} = \Delta\mathrm{D}x; \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

from which, if x is calculated for $\mathrm{D}=0{\cdot}00015$, the resulting error in t involving $\Delta\mathrm{D}^2$ will be found to be very small, and only one table of x will be required for all normal thermometers. To find the accuracy required in tabulating x, if t is required to $\pm 1/1000$ degree, let $dt=0{\cdot}0005°$ and Δx be the permissible error in x.

From (22)

$$\Delta x = \frac{0{\cdot}0005}{\Delta\mathrm{D}\mathrm{X}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and if $\Delta\mathrm{D}$ has the maximum value of 2×10^{-6} and $t'=600$, then Δx will be $\pm 5{\cdot}5$, and the temperature intervals, at which this function should be tabulated, can be found for any part of the scale.

(ii.) *The Correction as an Addition to pt.*

At the time that Heycock and Neville put forward their scale correction, for small variations in D, as a coefficient of $t-pt$ depending only on D, it was apparently not noticed that if the resulting quantity

$$(t-pt)\frac{\Delta D}{0{\cdot}00015}$$

were added to pt, and a new value of $(t-pt)$ obtained from the table at pt', the resulting sum would give almost the exact value of t.

This can be shown as follows :—

As before, $X=(t'-pt)$ when $D=0{\cdot}00015$, and

$$t = pt+X+\frac{\Delta D dt}{dD}, \quad . \quad . \quad . \quad . \quad . \quad (24)$$

To find the change in pt, say Δpt, equivalent to the change in t of

$$\Delta D\frac{dD}{dt},$$

multiply this quantity by dpt/dt,

$$\Delta pt = \Delta D\frac{dpt}{dD}.$$

From the difference formula (7)

$$\frac{dpt}{dD} = -(t-100)t \text{ at constant } t,$$

and

$$t = (pt+\Delta pt)+X'$$

$$= [pt+\Delta D(t-100)t]+X', \quad . \quad . \quad . \quad (25)$$

where X' is $(t-pt)$ from the table at $(pt+\Delta pt)$.

In practice $(t-100)t$ is not known, but X has been found at pt, and for $(t-100)t$ we may write $X/0{\cdot}00015$, and the expression becomes

$$t = \left(pt+\Delta D\frac{1}{0{\cdot}00015}X\right)+X' \quad \text{(see (16))}, \quad . \quad (26)$$

and t will be accurate to 1/1000 degree except at the highest temperature.

The error in the value of t, obtained by this expression (26) may be examined as follows. Let Δt be the difference in t due to ΔD and writing T for the expression $(t-100)t$.

From (25)

$$\Delta t = \frac{dt}{dpt}\Delta \mathrm{DT},$$

where

$$\frac{dt}{dpt} = \frac{1}{1-0{\cdot}00015(2t-100)}.$$

From (26)

$$\Delta t' = \frac{dt}{dpt}\Delta \mathrm{DT}',$$

and the required error is

$$\Delta t - \Delta t' = \frac{dt}{dpt}\Delta \mathrm{D}(\mathrm{T}-\mathrm{T}'). \quad . \quad . \quad . \quad (27)$$

To evaluate this expression $(\mathrm{T}-\mathrm{T}')$ must be found in terms of t and D.

$$t = t' + \Delta \mathrm{D}\frac{dt}{d\mathrm{D}} \quad \text{by definition.}$$

$$\therefore \quad \mathrm{T} = t'^2 + \Delta \mathrm{D}\frac{dt}{d\mathrm{D}}\left(2t' + \Delta \mathrm{D}\frac{dt}{d\mathrm{D}} - 100\right) + 100t'$$

$$= \mathrm{T}' + \Delta \mathrm{D}\frac{dt}{d\mathrm{D}}\left(2t' + \Delta \mathrm{D}\frac{dt}{d\mathrm{D}} - 100\right).$$

$$\therefore \quad \mathrm{T}-\mathrm{T}' = \Delta \mathrm{D}\frac{dt}{d\mathrm{D}}(2t'-100) \quad \text{almost exactly.}$$

$$\mathrm{T}-\mathrm{T}' = \Delta \mathrm{D}\frac{d\mathrm{T}'}{d\mathrm{D}}.$$

From which the error

$$\Delta t - \Delta t' = \frac{dt}{dpt}(\Delta \mathrm{D})^2\frac{d\mathrm{T}'}{d\mathrm{D}}. \quad . \quad . \quad . \quad . \quad (27')$$

And from (11)

$$\frac{d\mathrm{T}'}{d\mathrm{D}} = \frac{dt}{dpt}\mathrm{T}(2t'-100).$$

The error may be written

$$\Delta t - \Delta t' = \left(\frac{dt}{dpt}\right)^2 \mathrm{T}(2t'-100)(\Delta \mathrm{D})^2, \quad . \quad . \quad (27'')$$

which is extremely small. It only amounts to $\pm 1/1000$ degree when $t=600$, and $\Delta \mathrm{D}$ is 2×10^{-6}. As this method of correcting $t-pt$ for small changes in D does not require any tabulated coefficient, it is probably the most convenient of the three methods given.

In order to avoid looking up two values of $t-pt$, from the table, it is only necessary to multiply the addition to pt

$$\Delta DX\frac{1}{0{\cdot}00015} \text{ by } \frac{dt}{dpt} \text{ or } \left(1+\frac{dX}{dpt}\right),$$

where dX/dpt is taken directly from the tables at pt, and sum as follows :

$$t = pt+X\left[1+\Delta D\left(1+\frac{dX}{dpt}\right)\frac{1}{0{\cdot}00015}\right]. \quad . \quad (28)$$

There are now three expressions for t, in terms of R_t-R_0, which, with the aid of a slide-rule and the tables aforementioned, will give t to $\pm 1/1000$ degree on the international temperature scale.

Where $pt = (R_t-R_0)(1+I)$: . . . . . See (3)

(1) $t = pt+X+(D-0{\cdot}00015)F$,

(2) $t = pt+X[1+(D-0{\cdot}00015)x]$, . . See (16)

(3) $t = \left(pt+\frac{(D-0{\cdot}00015)X}{0{\cdot}00015}\right)+X'$. . See (26)

A table of the values of x can be made out in a very short time at every 20°, and will be found sufficiently accurate for all normal thermometers.

The following table gives a few values of

$$D(t-100)t-0{\cdot}00015(t'-100)t', \text{ or } dt/dD,$$

when

$$D = 0{\cdot}000151 \text{ or } \Delta D = 10^{-6}:$$

$t°=$	200°	300°	400°	500°	600°
$10^{-6}F=$	0·021°	0·065°	0·134°	0·232°	0·360°

(d) *Reproduction of the International Scale by Different Thermometers.*

Having discussed the relation between the platinum scale of temperature and the "international scale," an example will be considered where two different platinum thermometers are used to determine an identical temperature. It is naturally desirable to use a temperature which lies at some distance from the fixed points required for determining the constants of the thermometer. In this case the benzophenone point was used.

Thermometer constants determined from observations at the ice and sulphur points:—

Thermometer A ..... I=0·00493 D=0·00014990

Thermometer B ..... I=0·00512 D=0·00015154

It will be seen that D for thermometer B is more than 1 per cent. above normal.

Date of observation	November 7th, 1930.		July 29th, 1930.	
Thermometer	A.	B.	A.	B.
Barometer, corrected	763·20 mm.	763·20	754·70	754·70
pt, observed	296·570	296·475	296·067	295·967
$(t-pt)$, from table	9·457	9·449	9·416	9·407
ΔDF	−0·007	+0·105	−0·007	+0·105
$t°$ C.	306·034	306·029	305·476	305·479
Same, Ct^d to 760 mm. ...	305·832	305·827	305·811	305·814

The small difference of 0·02° between the values at the two dates is probably a real difference, as the benzophenone used was from separate sources. The agreement shown by the two thermometers is remarkably good; in fact, the agreement in these cases must be, to some extent, accidental, as it is within the error of a single observation. The latter is taken as ±0·005°, with the apparatus used for the above example, and for boiling-point observation with one thermometer on the same day. Thermometer A has been in occasional use, up to temperatures of 600°, for some 30 years, without adjustment, whereas thermometer B has only recently been made up and used for comparative purposes.

The examples given show the remarkable extent to which the international scale of temperature can be reproduced by different platinum thermometers, even if the quality of the platinum in each is markedly different. The agreement depends on the fact that expression (1) represents the actual change of resistance of a platinum wire, with increase in temperature on the "gas" scale between 0 and 450°. A great many observations show that the scale is reproducible (by different standard thermometers) by relation (1) from 0 to 600°, within the normal errors of observation, at the different temperatures.

(iii.) *The Normal Error of a Single Observation.*

It is generally agreed that, with pure sulphur in the standard boiling-point apparatus, the "boiling" temperature can be reproduced with an accuracy of $\pm 1/100$ degree *. The error of determining the fundamental interval will depend upon the fineness with which small changes of resistance can be measured, and also the accuracy with which the atmospheric pressure is known. These will, of course, vary with different sets of apparatus, barometers, etc. But in general it may be said that the error here may be kept down to $\pm 0{\cdot}005°$, especially if a mean of several independent observations is used †. We can now examine the magnitude of the effect of these two errors at different temperatures.

The difference from "international" scale, resulting from an error of $\pm 0{\cdot}01°$ at the sulphur point, and an error of $-0{\cdot}005°$ in the value of the fundamental interval, can be found as follows :—

First take the ice-point error : from the expression

$$t = (R_t - R_0)(1+I) + D(t-100)t, \quad . \quad . \quad . \quad (a)$$

$$\frac{dt}{dI} = \frac{R_t - R_0}{1 - D(2t-100)}, \quad . \quad . \quad . \quad . \quad . \quad (b)$$

and the error resulting from the ice point *alone* can be evaluated at any point in the scale.

As D has been chosen from a sulphur-point observation, assume that this observation is at the correct temperature. Hence ΔI, the ice-point error mentioned above, is balanced by $\Delta' D$, the resulting error in D, at t_s.

$$\Delta I \frac{dt}{dI} + \Delta' D \frac{dt}{dD} = 0 \quad . \quad . \quad . \quad . \quad . \quad (c)$$

when $t = t_s$ (sulphur boiling-point).

* The normal error of the ice and sulphur points largely depend upon the accuracy with which the atmospheric pressure is known. If a $\frac{1}{2}$-in. Fortin type barometer is used, which has been certified against a standard, the error of any individual reading, combined with the remaining zero error of the instrument, will be at least $\pm 0{\cdot}05$ mm. and may be $\pm 0{\cdot}1$ mm. unless special care is taken. 0·1 mm. is equal to $\pm 0{\cdot}01°$ at the sulphur point and $\pm 0{\cdot}005°$ at the ice point.

† At first sight it might appear that the "ice-point" error of $0{\cdot}005°$ must increase directly with the temperature, in which case it would appear very important at the higher temperatures. Actually this does not happen, as the value of D is determined from the value of I used.

ΔI has been taken $+0{\cdot}00005$, and from the above expression the resulting error in D of Δ'D can be found.

$$\frac{5(R_t-R_0)10^{-5}}{1-D(2t-100)} = -\frac{(t-100)t\Delta'D}{1-D(2t-100)}$$

when $t=444{\cdot}600$.

$$\Delta'D = -\frac{5(R_t-R_0)10^{-5}}{(t-100)t}.$$

In this case

$$\Delta'D = -1{\cdot}376-10^{-7}. \quad . \quad . \quad . \quad . \quad . \quad (d)$$

Here an approximate value of R_t-R_0 can be used, by assuming $R_t-R_0=pt=t-(t-pt)$ from the tables, $=421{\cdot}60$, and the error resulting from an error of $+0{\cdot}005°$ at the ice point can be evaluated at any point in the scale.

Let this error be Δt_i, then

$$\Delta t_i = \Delta I\frac{dt}{dI}+\Delta'D\frac{dt}{dD}. \quad . \quad . \quad . \quad . \quad . \quad (e)$$

The error of $\pm 0{\cdot}01$ at the sulphur point has now to be considered. From (11), the resulting error in D, due to say $+0{\cdot}01°$ at the sulphur point, is at once found. Let this be Δ''D.

$$\Delta''D = \frac{dt[1-D(2t-100)]}{(t-100)t} = +0{\cdot}58\times 10^{-7}. \quad . \quad (f)$$

Then the two errors in D, as determined at the sulphur point, have been evaluated:

(d) due to an error of $+0{\cdot}005°$, at the ice point,

$$\Delta'D = -1{\cdot}376\times 10^{-7};$$

(f) due to an error of $+0{\cdot}01°$, at the sulphur point,

$$\Delta''D = +0{\cdot}58\times 10^{-7}.$$

The total difference from the international scale as defined by expression (1) can then be found at any temperature, resulting from an ice-point error of $+0{\cdot}005°$ and a sulphur-point error of $+0{\cdot}01°$. Let ΔI = the error in I, $-\Delta'$D the resulting error in D, and Δ''D = the error in D caused by the sulphur-point error of $+0{\cdot}01°$. Then the required scale error Δt becomes

$$\Delta t = \Delta I\frac{dt}{dI}+\frac{dt}{dD}(\Delta''D-\Delta'D). \quad . \quad . \quad . \quad (g)$$

To evaluate for the errors chosen from (*b*) and (11),

$$\Delta t = 10^{-5}\left[\frac{5(R_t - R_0) - 0{\cdot}008(t-100)t}{1-D(2t-100)}\right]. \quad . \quad (h)$$

As in finding $\Delta' D$, an approximate value of $R_t - R_0$ at $t°$ can be used from the table of $t - pt$.

In the above example, assuming $D = 0{\cdot}00015$, Δt has the following values:—

$t =$	200°	300°	400°	500°
$\Delta t =$	+0·0087°	+0·0105°	+0·0106°	+0·0087°

From the ice-point error alone,

$\Delta t_i =$	+0·0074	+0·0068	+0·0029	−0·0046

From the sulphur error alone,

$\Delta t_s =$	+0·0012	+0·0037	+0·0077	+0·0135

and should the sulphur-point error have been −0·01° instead of +0·01 as taken, the total error would have been considerably greater at the higher temperatures, but much less below 400°.

It may be noted that if the thermometer is arranged to read differentially, with a zero thermometer kept at a constant known temperature, as recommended by Prof. H. L. Callendar, the "ice-point" errors will be reduced in the proportion $(t - t_z/t)$, where t_z is the zero temperature.

Having analyzed the errors in the final value of t, resulting from the uncertainty of the reduction constants, how nearly the international scale can be reached, by a single observation of a temperature with a standard platinum thermometer, can next be considered. The accuracy of a resistance measurement cannot be indefinitely extended, when the thermometer is at high temperature, for the electrical connexions thereto are at slightly different and minutely variable temperatures. Very small thermoelectric effects become troublesome when an attempt is made to measure R to 1/10,000 ohm. With the type of resistance measuring apparatus, in general use with platinum thermometers and under working conditions, R may be measured to $\pm 1/5000$ ohm, and the difference $R_t - R_0$ to $\pm 1/2000$, if R_t is at a very steady temperature. By means of special apparatus $R_t - R_0$ may be obtained to 1/10,000 ohm or 0·001°, if $R_{100} - R_0 = 10$ ohm, at any rate for temperatures below 200°. However, the extra complication and time involved in improving the accuracy of the measurement of $R_t - R_0$ from 0·005° to 0·001° precludes its use, except for special researches, where a few *comparative* temperatures are required with great accuracy.

Only the relative errors, of the various coils of the bridge, or potentiometer, used for measuring the resistance, will enter into the value of $R_t - R_0$, because the constants of the thermometer in question will have been obtained with the same apparatus. With careful calibration of the various coils and if the instrument is kept at a constant temperature, the relative errors should be less than 1/10,000 ohm.

Returning now to a single observation of temperature, where exceptional precautions will be taken to render the temperature as nearly steady as possible at the desired moment, the value of $R_t - R_0$ may be obtained to $\pm 1/2000$. To this error must be added (1) the temperature correction to normal for the coils, (2) the above-mentioned relative errors of the resistance coils, (3) and the minute changes in the value of R_0 which may occur during the temperature observation, which may be evaluated approximately as follows:—

(1) Temperature correction ...	$\pm 1/10000$ ohm
(2) Relative coil error	$\pm 1/10000$ „
(3) Zero change.....	$\pm 1/5000$ „
Total	$\pm 0{\cdot}0004$ ohm.

Taking the case of a thermometer with a fundamental interval of 10 ohm, this total error $= \pm 0{\cdot}004°$. Adding this to the error of observing $R_t - R_0$, a total of $\pm 0{\cdot}01°$ in a single temperature observation is obtained. It has already been shown that the error resulting from the uncertainty of the thermometer constants averages $\pm 0{\cdot}01°$ from 200° to 500°. Thus the uncertainty of reproduction of a temperature on the international scale from 200° to 500° by two independent observers, using standard platinum thermometers, is $\pm 1/50°$ C., and this is in accord with the experience of many observers in different parts of the world.

At temperatures of 550 to 600° the uncertainty may be as large as $\pm 1/20°$.

Although the above total error may seem too large to the individual observer, who is constantly able to reproduce his temperatures within half this amount, even when he uses different thermometers, experience has shown that, when comparing the work of independent observers, a possible difference equal to the above total error must be taken into account for this part of the scale.

October 29th, 1931.

LXXX. *The Refraction of Gaseous Acetaldehyde and Ethyl Acetate. By* H. Lowery, *Ph.D., F.Inst.P., Head of the Department of Pure and Applied Physics,* H. C. Poole, *M.Sc., and* C. Isherwood, *B.Sc., College of Technology, Manchester* *.

Abstract.

THE gaseous refractive index of acetaldehyde in relation to the density of the substance is found to be 1·0007879 for λ 5461, while the dispersion over the range λ 4358–λ 6708 is represented by the Sellmeier expression $(\mu-1)_d = 7{\cdot}4653\times10^{27}/(9779{\cdot}3\times10^{27}-\nu^2)$, ν being the frequency of the incident light.

The gaseous refractive indices of ethyl acetate for λ 5461 and λ 5893 (N.T.P. conditions) are found to be 1·001408 and 1·001399 respectively, both results being in agreement with previous work of Mascart.

I. *Acetaldehyde.*

Only one determination of the refractive index of gaseous acetaldehyde has been previously recorded, namely, that by Mascart †. He compared the refractivity of the substance with that of air and found the ratio to be 2·76. From this result Dufet ‡ has reduced the refractive index of gaseous acetaldehyde under N.T.P. conditions to the value 1·000806 for sodium light. The dispersion has not been previously determined.

The apparatus employed in the present determination consisted of a Jamin interferometer and Hilger monochromator with a K. B. B. mercury-vapour lamp as source of light. Vapour from a reservoir containing liquid acetaldehyde was allowed to enter one of the refraction tubes gradually, the other tube remaining permanently evacuated throughout the experiments, and the number of interference fringes displaced by a determined amount of vapour was obtained. A density bulb connected to the refraction tube into which the vapour passed served to enable the amount of vapour producing retardation of the light to be found.

* Communicated by the Authors.
† *Comptes Rendus,* vol. lxxxvi. p. 1182 (1878).
‡ ' Recueil de Données Numériques ' (1898).

The following are the results of ten determinations of the refractivity for the green mercury line, λ 5461 :—

$(\mu-1)_d \times 10^7$	7869	7883	7875	7878	7885
	7875	7885	7888	7874	7877
Mean.....	7879				

where $(\mu-1)_d$ represents the refractivity in relation to the density, *i. e.*, the results show the refractivity of gaseous acetaldehyde by the same number of molecules as 1 cm.^{3} of hydrogen contains at N.T.P.

The data employed in reducing the observations were :— Molecular weight=44·03 ; standard density=1·962 gm./litre ; mean experimental density for N.T.P. conditions= 2·026 gm./litre.

Assuming the above value for the refractive index as determined for λ 5461, the dispersion was measured over the range λ 4358–λ 6708 with the results shown in the table.

Dispersion of Gaseous Acetaldehyde.

	$(\mu-1)_d \times 10^7$.		Difference.
	Observed.	Calculated.	
Li 6708	7791·8	7793·2	+1·4
Cd 6438	7806·3	7807·1	+0·8
Li 6104	7827·4	7827·1	−0·3
Hg 5770	7853·4	7850·8	−2·6
Hg 5461	7878·9	7876·9	−2·0
Cd 5086	7915·8	7915·4	−0·4
Cd 4800	7850·5	7951·4	+0·9
Li 4602	7979·5	7980·6	+1·1
Hg 4358	8022·3	8022·5	+0·2

The observed values of the second column lead to the Sellmeier equation ;—

$$(\mu-1)_d = 7{\cdot}4653 \times 10^{27}/(9779{\cdot}3 \times 10^{27} - \nu^2),$$

where ν is the frequency of the incident light, and the figures of the third column have been calculated from this equation.

Comparison of the present results with the determination of Mascart may be effected thus :—the Sellmeier equation gives the value of $(\mu-1)_d$ for sodium light as

0·0007841. On altering this in the ratio : (Experimental density at N.T.P.)/(Standard density)=2·2026/1·962, the value of ($\mu-1$) under N.T.P. conditions is found to be 0·0008097. This corresponds to Mascart's value of 0·000806 as reduced by Dufet *.

Pure liquid acetaldehyde for the experiments was obtained from Messrs. British Drug Houses, Limited.

II. *Ethyl Acetate.*

Both Mascart † and Lorenz ‡ have made observations on the refractive index of gaseous ethyl acetate, but their values for sodium light differ sufficiently to call for further determinations, thus Mascart obtained 1·001400 while the result of Lorenz is 1·001586 for N.T.P. conditions. Lorenz obtained also the result 1·001578 for λ 6708.

Eleven determinations of the refractivity for λ 5461 gave the following values :—

$(\mu-1)_{N.T.P.}\times 10^6$....	1401	1411	1415	1414,	1409
	1406	1411	1406	1415,	1401
	1401				
Mean.....	1408				

An independent set of six experiments made previously by one of us (H. L.) for *sodium light* gave as a mean value :—

$$(\mu-1)_{N.T.P.}= 0{\cdot}001399.$$

It will be seen, therefore, that both sets of experiments support the results of Mascart.

LXXXI. *Note on Dr. Wheeler's Discussion of the Madelung Constants for some Cubic Crystal Lattices.* *By* J. Sherman, *Ph.D.* §

IN the July issue of the Phil. Mag. for 1932, Dr. T. S. Wheeler has developed a simple method for the calculation of the electrostatic potential of a simple cubic

* *Loc. cit.*
† *Loc. cit.*
‡ *Wien. Ann.* vol. xi. p. 99 (1880).
§ Communicated by the Author.

lattice with respect to a lattice point and with respect to a point at the centre of one of the cubes, and has applied these results to the calculation of an approximate value for the Madelung constant of the cæsium chloride structure. Combining this value for the cæsium chloride structure with the Madelung constant of the sodium chloride structure obtained from earlier treatments, he has calculated the Madelung constants of the fluorite and zinc blend structures. Finally, he has reported a value for the Madelung constant of the cuprite structure obtained from that for fluorite; the value which he gives for cuprite, 3·162, is, however, incorrect, the correct result, calculated by Emersleben, being 4·7522.

The following considerations show the error in Dr. Wheeler's calculation. The coulomb energy of the copper ions in cuprite with respect to a copper ion is one-fourth of the energy of the calcium ions in fluorite with respect to a calcium ion. The coulomb energy of the copper ions with respect to an oxygen ion is the same as the energy of the calcium ions with respect to a fluorine ion. However, the coulomb energy of the oxygen ions in cuprite with respect to an oxygen ion is that of a body-centred lattice with respect to a lattice point, whereas the coulomb energy of the fluorine ions in fluorite with respect to a fluorine ion is that of a simple cubic lattice with respect to a lattice point. Hence in evaluating the Madelung constant of the cuprite structure from that of fluorite, the difference between these two energies must be taken into account, as has been done by Dr. Wheeler. Finally, the coulomb energy of the oxygen ions with respect to a copper ion is that of a body-centred lattice with respect to an ion at $\frac{1}{4}\frac{1}{4}\frac{1}{4}$, whereas the coulomb energy of the fluorine ions with respect to a calcium ion is that of a simple cubic lattice with respect to an ion at the centre of one of the cubes, and the difference between these two energies must also be taken into account. This Dr. Wheeler has failed to do, the error in the value of the Madelung constant which he reports being due to just this difference.

California Institute of Technology,
Pasadena, California.
August 13, 1932.

LXXXII. *The Distribution of Sound Radiation from a Sphere vibrating in various Ways: with Applications to Loud Speaker Diaphragms.* By N. W. McLachlan, *D.Sc.*, *M.I.E.E.* *

Synopsis.

1. Introduction: Derivation of Formulæ.
2. Sphere vibrating axially.
3. One Hemisphere vibrating axially, the other quiescent.
4. Two Hemispheres vibrating axially in opposition.
5. Sphere with n Nodal Circles.

Abstract.

By aid of Legendre functions formulæ are deduced for the acoustic pressure at a great distance from a sphere vibrating in various ways. Polar diagrams are given showing the distribution of radiation for (*a*) sphere vibrating axially, (*b*) hemisphere vibrating axially, (*c*) two hemispheres vibrating axially in opposition, (*d*) sphere with two nodal circles. The results are applied to hornless loud speakers.

1. *Introduction: Derivation of Formulæ.*

When a sphere vibrates in an infinite mass of fluid the condition to be satisfied at every point therein is

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}=\frac{1}{c^2}\frac{d^2\phi}{dt^2}, \quad . \quad . \quad . \quad . \quad (1)$$

where ϕ is the velocity potential at the point whose co-ordinates, referred to the centre of the sphere, are x, y, z, and c is the velocity of longitudinal waves in the fluid.

If the motion of the sphere obeys a sine wave law we can write

$$\phi=\phi_1\cos\omega t,$$

from which we obtain

$$\frac{d^2\phi}{dt^2}=-\omega^2\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting for $\frac{d^2\phi}{dt^2}$ from (2) in (1) we have

$$(\nabla^2+k^2)\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

* Communicated by the Author.

where

$$k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

and

$$\nabla^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}.$$

Since we are concerned with the vibration of a sphere it is essential to transform equation (1) to spherical co-ordinates. Writing

$$x = r\sin\theta\cos\chi,$$
$$y = r\sin\theta\sin\chi,$$
$$z = r\cos\theta$$

Fig. 1.

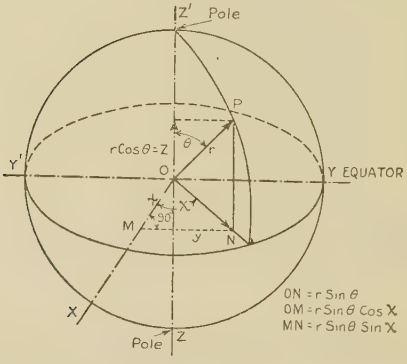


Diagram showing orientation of axes, also the angles θ and χ.
$x = r\sin\theta\cos\chi$, $y = r\sin\theta\sin\chi$, $z = r\cos\theta$, r = radius of sphere.

(fig. 1), and performing the requisite algebraic operations, we ultimately obtain the well-known differential equation* of spherical harmonics, viz.,

$$r^2\frac{d^2\phi}{dr^2} + 2r\frac{d\phi}{dr} + \frac{1}{\sin\theta}\cdot\frac{d}{d\theta}\left(\sin\theta\frac{d\phi}{d\theta}\right) + \frac{1}{\sin^2\theta}\cdot\frac{d^2\phi}{d\chi^2} + k^2r^2\phi = 0. \quad \ldots \quad (4)$$

The solution to this equation is of the form

$$\phi = \phi_0 + \phi_1 + \ldots\phi_n,$$

where ϕ_n is associated with a spherical harmonic of the nth order. Each quantity ϕ_n is separately a solution of (4).

* See, for example, 'Spherical Harmonics' by T. M. MacRobert.

For disturbances propagated wholly *outwards* from the sphere

$$\phi_n = \frac{1}{r} \cdot S_n(\mu, \chi) e^{-z} f_n(z)^*, \quad . \quad . \quad . \quad . \quad (5)$$

where $S_n(\mu, \chi)$ is the general spherical harmonic,

$z = ikr$, $\quad i = \sqrt{-1}$,

r = distance of point from centre of sphere,

$\mu = \cos\theta$,

and

$$f_n(z) = 1 + \frac{n(n+1)}{2} z^{-1} + \frac{(n-1)\ldots(n+2)}{2 \cdot 4} z^{-2}$$

$$+ \ldots \frac{1 \cdot 2 \cdot 3 \ldots 2n}{2 \cdot 4 \ldots 2n} \cdot z^{-n}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

By differentiating (5) with respect to r we get the particle velocity due to the nth harmonic, namely,

$$v_n = \frac{d\phi_n}{dr}.$$

$$v_n = \frac{d\phi_n}{dr} = -\frac{1}{r^2} S_n(\mu, \chi)\{e^{-z} F_n(z)\}, \quad . \quad . \quad (7)$$

where

$$F_n(z) = (1+z) f_n(z) + z f_n'(z).$$

At the surface of the sphere the radial velocity u_n due to the nth harmonic is equal to the particle velocity, *i. e.*, $u_n = v_n$ and $r = a$. Thus from (7) we find that

$$u_n = -\frac{S_n(\mu, \chi)}{a^2} \cdot e^{-ika} \cdot F_n(ika),$$

or

$$S_n(\mu, \chi) = -\frac{a^2 u_n e^{ika}}{F_n(ika)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Substituting the above value of $S_n(\mu, \chi)$ in (5), we obtain

$$\phi_n = -\frac{a^2}{r} \cdot e^{ik(a-r)} \, u_n \frac{f_n(ikr)}{F_n(ika)}, \quad . \quad . \quad . \quad (9)$$

this being the velocity potential due to the nth harmonic at a distance r from the centre of the sphere.

At a great distance $f_n(ikr)$ is substantially unity, whilst the quantity $-\frac{a^2}{r} e^{ik(a-r)}$ is constant for any particular

* Rayleigh, 'Sound,' ii. p. 237 (1894).

frequency and independent of the order of the harmonic. Thus the distribution of sound is dependent upon

$$\mathscr{R} = \sum \frac{u_n}{\mathrm{F}_n(ika)} . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

We shall for convenience regard $\mathscr{R}$ as the spatial radiation characteristic.

The quantity $\frac{1}{\mathrm{F}_n(ika)}$ can be written $x_n + iy_n$, so that

$$\mathscr{R} = \mathrm{A} + i\mathrm{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$= (u_0x_0 + u_1x_1 + \ldots u_nx_n) + i(u_0y_0 + u_1y_1 + \ldots). \quad (12)$$

The value we require is the modulus $|\mathscr{R}| = \sqrt{\mathrm{A}^2 + \mathrm{B}^2}$.

TABLE I.

ka.	x_0.	y_0.	x_1.	y_1.	x_2.	y_2.	x_4.	y_4.
0	1·0	−0·0	0	0	0	0	0	0
0·5.....	0·8	−0·4	+0·125	+0·22	−0·024	+0·0135	...	...
1·0.....	0·5	−0·5	+0·2	+0·2	−0·056	+0·09	+0·001	−0·001
2·0..... ...	0·2	−0·4	−0·1	−0·2	+0·188	+0·268	...	...
3·0.....	0·1	−0·3	−0·082	−0·246	+0·33	0	...	...
4·0.....	0·059	−0·235	−0·054	−0·216	+0·2	−0·1	...	...
5·0.....	0·038	−0·192	−0·366	−0·183	+0·16	−0·128	...	...
10·0.....	0·01	−0·1	−0·01	−0·1	+0·04	−0·1	+0·1	−0·041

A series of values of x_n and y_n corresponding to a certain range of the quantity ka are given in Table I.

All the radial velocity components $u_0u_1\ldots$ are functions of the angles θ and χ (fig. 1), so that it is possible to calculate the separate terms in (12) and thence the radiation characteristic for various values of θ and χ.

When the vibration of the sphere is symmetrical about the polar axis ZOZ′ (fig. 1) the fourth term in (4) disappears, and the solution is then expressed in zonal surface harmonics, *i.e.*, Legendre functions. To obtain the radial velocity u of the sphere in the form of an expansion we write

$$u = \mathrm{A}_0\mathrm{P}_0(\mu) + \mathrm{A}_1\mathrm{P}_1(\mu) + \mathrm{A}_2\mathrm{P}_2(\mu)\ldots\mathrm{A}_n\mathrm{P}_n(\mu), \quad (13)$$

where $\mathrm{A}_1\ldots\mathrm{A}_n$ are constant coefficients, $\mathrm{P}_0(\mu)$, $\mathrm{P}_1(\mu)$, etc. are Legendre functions (zonal surface harmonics), and $\mu = \cos\theta$. Several of these functions are given in Table II.

Multiplying both sides of (13) by $P_n(\mu)\,d\mu$, and integrating from -1 to $+1$, we obtain the well-known formula

$$u_n = A_n P_n(\mu) = (n+\tfrac{1}{2}) P_n(\mu) \int u\, P_n(\mu)\, d\mu, \quad . \quad (14)$$

the limits of integration now being dependent upon the distribution of the radial velocity u over the surface of the sphere.

The solution of (4) when χ is involved will be treated later. It is in the form of sectorial harmonics.

TABLE II.—Legendre functions.

$$P_0(\mu) = 1,$$
$$P_1(\mu) = \mu,$$
$$P_2(\mu) = \tfrac{1}{2}(3\mu^2 - 1),$$
$$P_3(\mu) = \tfrac{1}{2}(5\mu^3 - 3\mu),$$
$$P_4(\mu) = \tfrac{1}{8}(35\mu^4 - 30\mu^2 + 3).$$

2. *Sphere vibrating axially.*

The maximum axial velocity varying sinusoidally is U, so the radial velocity is $u = U\cos\theta = U\mu$ (fig. 2 *a*). This value of u, being proportional to the zonal harmonic of unit order—namely, $P_1(\mu) = \mu$,—it is unnecessary to obtain an expansion. Thus from (10) the radiation characteristic is given by

$$\mathscr{R} = \frac{u}{F_1(ika)} = \frac{U\mu}{F_1(ika)}. \quad . \quad . \quad . \quad . \quad (15)$$

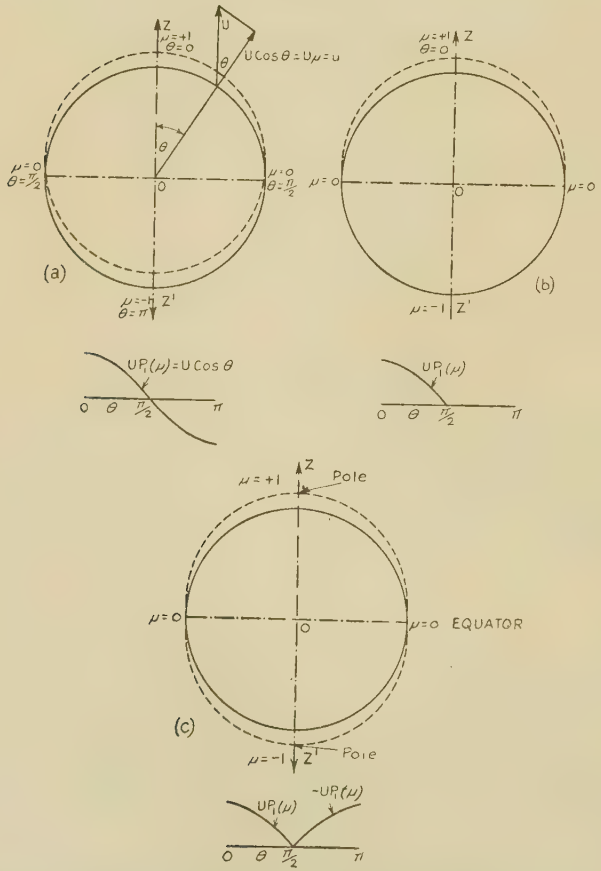
Since $\dfrac{U}{F_1(ika)}$ merely affects the magnitude of $\mathscr{R}$, the distribution of radiation depends upon $\mu = \cos\theta$, and it is obviously two circles, as shown in fig. 3 *a*. This form is preserved at all frequencies, although, of course, there is a variation in the power radiated throughout the frequency range.

From this we can obtain an approximate conception of the radiation from a loud speaker diaphragm vibrating as a whole without a baffle, or at low frequencies with a small baffle. A conical diaphragm is concave on one side and convex on the other, which causes a modification in the distribution from the two sides as compared with that for a sphere. However, we see that in the plane containing the periphery or base of the diaphragm there will be little or no sound.

3. *One Hemisphere vibrating axially, the other quiescent.*

The maximum axial velocity varying sinusoidally is U from $\mu=1$ to 0 and zero from $\mu=0$ to -1. Thus the radial

Fig. 2.



(*a*) Sphere vibrating axially as a whole. Radial velocity $u=U\mu$.

(*b*) Hemisphere vibrating axially as a whole. $u=U\mu$ from $\theta=0$ to $\pi/2$.

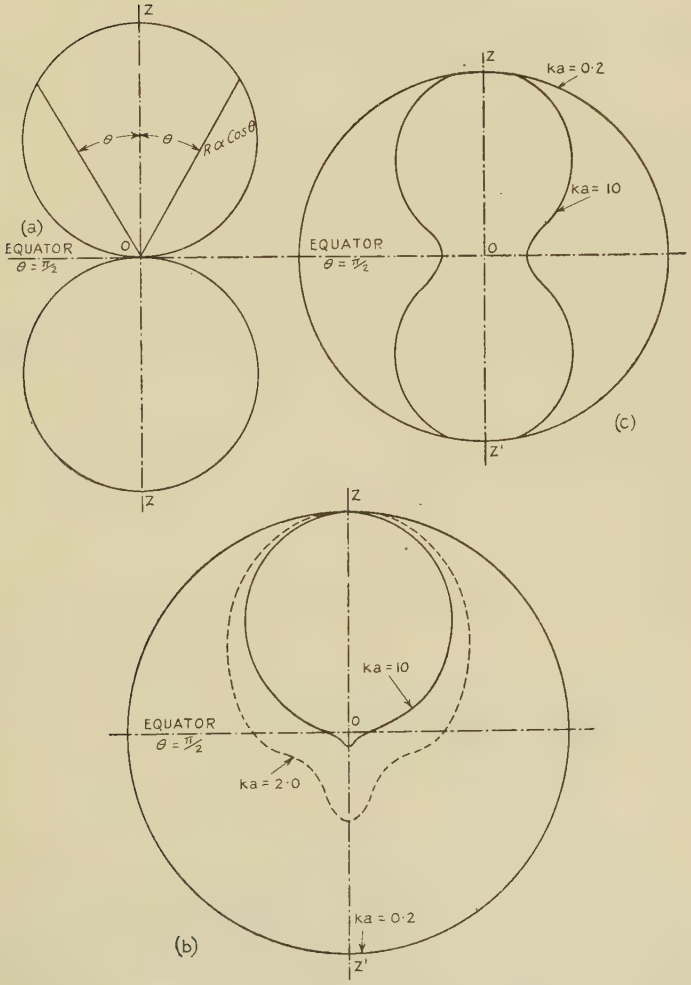
(*c*) Hemispheres vibrating axially in opposition. $u=+U\mu$ from θ to $\pi/2$, $-U\mu$ from $\pi/2$ to π.

velocities in the two hemispheres are respectively $u=U\mu$ and $u=0$ (fig. 2 *b*).

From (14) we have

$$u_n = (n+\tfrac{1}{2})\,\mathrm{P}_n(\mu)\,\mathrm{U}\int_0^1 \mu \mathrm{P}_n(\mu)\,d\mu + 0,$$

Fig. 3.



(*a*) Radiation characteristic of sphere vibrating axially; for all values of *ka*.

(*b*) Radiation characteristic of hemisphere vibrating axially; $ka = 0{\cdot}2$, $2{\cdot}0$, $10{\cdot}0$.

(*c*) Radiation characteristic of two hemispheres in opposition; $ka = 0{\cdot}2$, $10{\cdot}0$.

$$u_0 = \tfrac{1}{2}\mathrm{U}\int_0^1 \mu\, d\mu = \frac{\mathrm{U}}{4},$$

$$u_1 = \tfrac{3}{2}\mu\mathrm{U}\int_0^1 \mu^2 d\mu = \frac{\mathrm{U}}{2}\cdot\mu,$$

$$u_2 = \tfrac{5}{2}\mathrm{P}_2(\mu)\,\mathrm{U}\int_0^1 \tfrac{1}{2}(3\mu^3-\mu)\,d\mu$$

$$= \frac{5}{32}(3\mu^2-1)\,\mathrm{U},$$

$$u_3 = 0,$$

$$u_4 = \tfrac{9}{2}\mathrm{P}_4(\mu)\,\mathrm{U}\int_0^1 \tfrac{1}{8}(35\mu^5-30\mu^3+3\mu)\,d\mu$$

$$= -\frac{3}{256}(35\mu^4-30\mu^2+3)\mathrm{U}.$$

The contributions from higher harmonics can be neglected, since they are relatively unimportant. The radiation characteristic from (11) and (12) is consequently $\mathscr{R} = \mathrm{A}+i\mathrm{B}$, where

$$\mathrm{A} = \mathrm{U}\left\{\tfrac{1}{4}x_0+\tfrac{1}{2}\mu x_1+\frac{5}{32}(3\mu^2-1)x_2-\frac{3}{256}(35\mu^4-30\mu^2+3)x_4\right\}, \quad \ldots \quad (16)$$

and

$$\mathrm{B} = \mathrm{U}\left\{\tfrac{1}{4}y_0+\tfrac{1}{2}\mu y_1+\frac{5}{32}(3\mu^2-1)y_2-\frac{3}{256}(35\mu^4-30\mu^2+3)y_4\right\}. \quad \ldots \quad (17)$$

Using the values of x_0, x_1, y_0, y_1, etc. given in Table I., corresponding to $ka=10$, we find that

$$\mathrm{A} = \mathrm{U}(-0{\cdot}0073+0{\cdot}01\mu+0{\cdot}0538\mu^2-0{\cdot}041\mu^4), \quad (18)$$

and

$$\mathrm{B} = -\mathrm{U}(0{\cdot}0078+0{\cdot}05\mu+0{\cdot}062\mu^2-0{\cdot}0175\mu^4). \quad (19)$$

Inserting various values of μ corresponding to different angles θ in (18) and (19) the modulus of $\mathscr{R}$ can be found. This is portrayed graphically in fig. 3 *b*. The value of ka being 10, if we assume the sphere to be 10 cm. radius the corresponding value of the frequency is 5400~. Since the wave-length of sound is then comparable with the diameter of the sphere, appreciable focussing ensues and practically all the radiation is in front of the active hemisphere. The distribution is very nearly circular. At higher frequencies the same form is substantially preserved.

At low frequencies, *e. g.*, 108∼, where $ka=0{\cdot}2$, all the terms in (16) and (17) except that containing x_0 are negligible. Thus $\mathscr{R}\doteqdot\frac{1}{4}\mathrm{U}$, so that the distribution, being independent of μ, is identical everywhere, *i. e.*, it is spherical, as shown in fig. 3 *b*, $ka=0{\cdot}2$.

At an intermediate value where $k=2{\cdot}0$, $f=1080\sim$, and the polar curve takes the form illustrated in fig. 3 *b*. Here there is a proportion of the radiation behind the quiescent half of the sphere.

The results of the preceding analysis can be applied—with the necessary limitations—to a loud speaker diaphragm vibrating in a small absorbent cabinet. At low frequencies the distribution is spherical, there being sound behind as well as in front of the reproducer. As the frequency increases the sound behind the cabinet steadily decreases until at high frequencies it is inaudible—in free space or its equivalent—owing to the focussing or beam effect. The high-frequency output is then contained mainly in a comparatively small solid angle in front of the diaphragm.

4. *Two Hemispheres vibrating axially in opposition.*

The axial velocity is $+\mathrm{U}$ from $\mu=1$ to 0 and $-\mathrm{U}$ from $\mu=0$ to -1, so the corresponding radial velocities are $+\mathrm{U}\mu$ and $-\mathrm{U}\mu$ (fig. 2 *c*). Thus

$$u_n=(n+\tfrac{1}{2})\mathrm{P}_n(\mu)\mathrm{U}\left\{\int_0^1\mu\mathrm{P}_n(\mu)\,d\mu-\int_{-1}^0\mu\mathrm{P}_n(\mu)\,d\mu\right\}. \quad (20)$$

From fig. 2 *c* and the symmetry of the motion about the equatorial plane $\mathrm{Z}=0$ it is clear that the expansion of the radial velocity contains even harmonics only; whence we can write (20) in the form

$$u_n=(2n+1)\mathrm{P}_n(\mu)\mathrm{U}\int_0^1\mu\mathrm{P}_n(\mu)\,d\mu, \quad . \quad . \quad (21)$$

where n is even. Thus

$$u_0=\mathrm{U}\int_0^1\mu\,d\mu=\frac{\mathrm{U}}{2},$$

$$u_2=5\mathrm{P}_2(\mu)\mathrm{U}\int_0^1\tfrac{1}{2}(3\mu^3-\mu)\,d\mu$$

$$=\frac{5}{16}(3\mu^2-1)\mathrm{U},$$

$$u_4=\frac{9}{2}\mathrm{P}_4(\mu)\mathrm{U}\int_0^1\tfrac{1}{8}(35\mu^5-30\mu^3+3\mu)\,d\mu$$

$$=-\frac{3}{128}(35\mu^4-30\mu^2+3)\mathrm{U}.$$

Then $\mathscr{R}=A+iB$, where

$$A = U\left\{\tfrac{1}{2}x_0+\frac{5}{16}(3\mu^2-1)x_2-\frac{3}{128}(35\mu^4-30\mu^2+3)x_4\right\} \quad . \quad . \quad . \quad (22)$$

and

$$B = U\left\{\tfrac{1}{2}y_0+\frac{5}{16}(3\mu^2-1)y_2-\frac{3}{128}(35\mu^4-30\mu^2+3)y_4\right\}. \quad . \quad . \quad (23)$$

Using the values of x_0, x_1, y_0, y_1, etc. given in Table I., corresponding to $ka=10, f=5400\sim$, we obtain

$$A = U\{-0{\cdot}0145+0{\cdot}108\mu^2-0{\cdot}08\mu^4\} \quad . \quad . \quad (24)$$

and

$$= U\{0{\cdot}016+0{\cdot}124\mu^2-0{\cdot}035\mu^4\}. \quad . \quad . \quad (25)$$

The value of the modulus $\sqrt{A^2+B^2}$ is now calculated for various angles θ, and the results are depicted in fig. 3 *c*, $ka = 10$. Owing to the interference each hemisphere experiences due to its own radiation the sound tends to be propagated in beam formation. This accounts for the appreciable reduction in output as the equatorial plane is approached. The distribution of radiation in *each hemisphere* is almost identical with that of a single hemisphere acting alone, as shown in fig. 3 *b*, $ka=10$. Ultimately, as ka increases the polar diagram undergoes a slight modification and the radiation at $\theta=90°$ ($\mu=0$) may vanish. To test this it would be necessary to incorporate higher harmonics in the expansion of the radial velocity.

As in the preceding case, when ka is 0·2, all the terms in (22) and (23) except x_0 are negligible, and the distribution at low frequencies is spherical (fig. 3 *c*).

This example can be compared roughly with that of a double loud speaker where two diaphragms moving axially in opposition are employed*. The radiation from the inner faces of the diaphragms is suppressed, to avoid interference with that from the outer faces. The result is to yield spherical distribution at low frequencies. It is possible therefore to dispense with a baffle. As the frequency rises the usual focussing effect from the outer sides of the diaphragms ensues. In the average room the distribution is levelled up to an extent owing to reflexion.

* See Patent Specifications 271, 021 and 270, 421, due to the author.

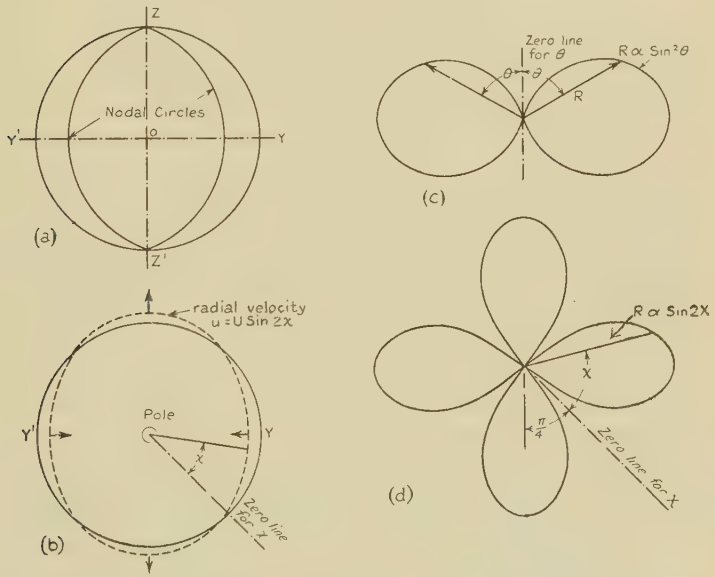
5. *Sphere with n Nodal Circles.*

The motion in this instance being asymmetrical about the polar axis ZOZ′ (figs. 1 and 4 *a*) the solution of equation (4) is now in terms of sectorial harmonics of the form

$$T_n^n(\mu) = K \sin^n\theta \sin n\chi,$$

where K is a coefficient * depending upon n.

Fig. 4.



(*a*) Diagram showing sphere vibrating with two nodal circles.

(*b*) Section of (*a*) on equatorial plane Y′OY. The zero line for the angle χ has been moved counter-clockwise through $\pi/4$ for convenience.

(*c*) Distribution of radiation on any plane containing axis ZOZ′.

(*d*) Distribution of radiation on any plane parallel to the equatorial plane Z=0.

We shall assume that the radial velocity is given by

$$u = U \sin^n\theta \sin n\chi. \quad . \quad . \quad . \quad . \quad . \quad (26)$$

From formula (10) we have the radiation characteristic

$$\mathscr{R} = \frac{U \sin^n\theta \sin n\chi}{F^n(ika)}, \quad . \quad . \quad . \quad . \quad (27)$$

an expansion being unnecessary, since $U \propto T_n^n(\mu)$, this being

* $K = d^n P_n(\mu)/d\mu^n = 1.3.5.\ldots.(2n-1)$.

a solution of (4). Since $F_n(ika)$ is independent of θ and χ, it merely affects the intensity of the radiation at various frequencies. It follows therefore that the polar diagram is the same whatever the frequency or the radius of the sphere. Also the distribution of radiation is identical in shape with the deformation curve of the sphere *.

By way of illustration suppose we treat the case of two nodal circles, *i. e.*, $n=2$. Then the characteristic

$$\mathcal{R} \propto U \sin^2\theta \sin 2\chi. \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Referring to fig. 4, imagine a plane containing ZOZ′ to rotate abont this axis, the starting point being $\chi=\frac{\pi}{4}$. Inserting this value of χ in (28) we obtain

$$\mathcal{R} \propto U \sin^2\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

The distribution over the plane at $\chi=\frac{\pi}{4}$, as plotted from (29), is shown in fig. 4 *c*. As the plane rotates counter-clockwise about ZOZ′ the angle χ increases and $\sin 2\chi$ decreases. The distribution of radiation remains unaltered in shape but reduced in intensity according to a sine law until at $\chi=\frac{\pi}{2}$ it disappears altogether. Thereafter it grows in intensity according to a sine law until $\chi=\frac{3}{4}\pi$, when it is again a maximum. This cycle of events is repeated in each quadrant, as illustrated in fig. 4 *d*. In one rotation of the plane there are four maxima and four zero positions. When there are n nodal circles the maxima and zero positions number $2n$. Obviously there is no sound radiation on the polar axis ZOZ′.

This analysis gives a broad idea of the radiation distribution to be expected from a free edge conical diaphragm vibrating at low frequencies with nodal diameters in a large baffle.

May 1932.

LXXXIII. *The Kaufmann-Bucherer Experiments and a Classical Einsteinian Mass Formula.* *By* A. Press †.

1. THE problem of a moving electrically charged body has been treated by Oliver Heaviside, notably in his 'Electromagnetic Theory,' vol. i. pp. 269–274. The

* Stokes, Phil. Mag. xxxvi. p. 401 (1868).

† Communicated by the Author.

method of attack was to consider the charged body stationary in space and proceed on the line that the aether moved by relative to the body and to fixed space. One thing should have been emphasized, that is, if the body is negatively charged the counterpart or positive charge was contrariwise fixed in the moving aether medium supporting the electromagnetic generalized stresses and strains.

Using Heaviside's notation with c as the dielectric constant, the fundamental circuital laws still become

$$\text{curl}\,(\mathbf{H}-\mathbf{h})=cp\mathbf{E}+\rho\mathbf{u}=c\frac{d\mathbf{E}}{dt}+\rho\mathbf{u},$$

$$\text{curl}\,(\mathbf{e}-\mathbf{E})=\mu p\mathbf{H}=\mu\frac{d\mathbf{H}}{dt}. \quad . \quad . \quad . \quad . \quad . \quad (1.1)$$

But care has to be taken in interpreting $\mathbf{E}$ and $\mathbf{H}$ because of the positive charge moving with the aether. Thus, provided the motion has been going on sufficiently long to make the distance between charges sufficiently great, we can assume that the electrical intensity $\mathbf{E}$ near the non-moving charge will be substantially constant with respect to the time and correspond to a "point-charge" distribution (see *l. c.* iii. p. 432). That is, we have for the problem at hand that

$$\frac{d\mathbf{E}}{dt}\equiv 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.2)$$

It is this condition of the body carrying ρ that practically gives rise to a so-called "free" charge effect. Nevertheless no charge is strictly speaking free. Turning therefore to the first of equations (1.1), since ρ does not move relative to fixed space it means that the translational velocity $\mathbf{u}$ is zero, so that the entire right-hand side is zero, with the consequence that we have

$$\mathbf{H}\equiv\mathbf{h}=\mathrm{V}\mathbf{D}\mathbf{w}; \quad . \quad . \quad . \quad . \quad . \quad (1.3)$$

that is, whatever impressed magnetic intensity $\mathbf{H}$ conformable to $\mathbf{E}$ which occurs (as a result of the translatory velocity $\mathbf{w}$ of the medium past the density $\mathbf{D}$ emanating from ρ) has its magnitude defined by the motional equation

$$\mathrm{V}\mathbf{D}\mathbf{w}=\mathbf{h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.4)$$

Thus, if $\mathbf{D}$ and $\mathbf{w}$ do not vary with the time, then the right-hand side of the second equation of (1.1) will be zero also, and we have the addition to (1.3), namely the characterizing equation

$$\text{curl}\,(\mathbf{e}-\mathbf{E})\equiv 0. \quad . \quad . \quad . \quad . \quad . \quad (1.5)$$

2. Having reference to a negative charge which is stationary in space, it is possible to adopt with Heaviside the notation that the translational velocity **w** of the medium can be written in the form

$$\mathbf{w}=-\mathbf{u}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.1)$$

with the understanding that the present **u** is not the one of equations (1.1). In this manner for the second motional effect we have for the electrical intensity **e** set up by the moving medium past ρ that

$$\mathbf{e}=\mathrm{V}\mathbf{u}\mathbf{B}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.2)$$

The argument of (1.5) is therefore a vector resultant **f**, such that in view of (1.4) we have

$$\mathbf{f}=\mathbf{E}+\mu\mathrm{V}\mathbf{u}\mathrm{V}\mathbf{u}\mathbf{D}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.3)$$

Such a double vector product as is here contained really reduces to give to equation (2.3) the form

$$\mathbf{f}=\mathbf{E}+\mu cu^2(\mathbf{k}\mathrm{E}_3-\mathbf{E}), \quad . \quad . \quad . \quad . \quad (2.4)$$

where we have that

$$\begin{aligned} v&=(\mu c)^{-\cdot 5}=\text{velocity of light,}\\ \mathrm{E}_3&=\text{electrical intensity component}\\ &\quad\text{in direction of } \mathbf{u}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.5)\end{aligned}$$

On rewriting (2.4) we have that for a new vector **F** defined by

$$\left(1-\frac{u^2}{v^2}\right)\mathbf{F}=\mathbf{f}, \quad . \quad . \quad . \quad . \quad . \quad (2.6)$$

$$\mathbf{F}=\mathrm{E}_1\mathbf{i}+\mathrm{E}_2\mathbf{j}+\frac{\mathrm{E}_3}{1-\frac{u^2}{v^2}}\mathbf{k}. \quad . \quad . \quad . \quad . \quad (2.7)$$

Up to this point the interpretation is substantially that given by Heaviside except for the limitation with regard to the positive charge embedded in the medium already referred to above. In so far as **u** is negligible with respect to **v** we would have that **F** was exactly the vector **E** which obtained before any movement of the medium transpired. It is submitted there is nothing in equation (2.7) that changes the magnitudes of E_1, E_2, or E_3 of themselves. Effectively, however, the coefficient of **k** is modified because of the denominator $\left(1-\frac{\mathbf{u}^2}{v^2}\right)$. To this extent a new electromagnetic masking effect of the charge on the electron is deducible,

which it will be shown later leads to a classical type of Lorentzian electron having a fixed transverse diameter and an effectively "fore-shortened" diameter in the direction of motion **u**. Nevertheless the longitudinal diminution need not be real. Electrically, however, an oblate-spheroidal or disk-shaped distribution follows, but with diminished power so far as distant points are concerned. There is then not so much a "concentration" of charge equatorially, as there is a masked diminution of effective charge polarly. This therefore enables one classically to avoid the Einsteinian "mass" concept.

3. Inasmuch as the original **E** was non-circuital the vector **F** of (2.7) is likewise non-circuital, so that it can be regarded as eolotropically derivable as the gradient of a scalar function P, where

$$\mathbf{F}=\mathrm{F}_1\mathbf{i}+\mathrm{F}_2\mathbf{j}+\mathrm{F}_3\mathbf{k},$$

$$\mathrm{F}_1=\mathrm{E}_1=-\frac{\partial \mathrm{P}}{\partial x},\quad \mathrm{F}_2=\mathrm{E}_2=-\frac{\partial \mathrm{P}}{\partial y},\quad \mathrm{F}_3=\frac{\mathrm{E}_3}{1-\beta^2}=-\frac{\partial \mathrm{P}}{\partial z},$$

$$\beta^2=\frac{u^2}{v^2}. \qquad (3.1)$$

The eolotropic dielectric constants being respectively

$$c_1,\quad c_2,\quad c_3,$$

$$c_1=c_2=c, \qquad (3.2)$$

we can write

$$\mathrm{D}_1=c\mathrm{E}_1,\quad \mathrm{D}_2=c\mathrm{E}_2,\quad \mathrm{D}_3=c(1-\beta^2)\mathrm{E}_3. \qquad (3.3)$$

The vector equation of electrical displacement **D** thus becomes

$$\mathbf{D}=\mathrm{D}\mathbf{i}_1+\mathrm{D}_2\mathbf{j}+(1-\beta^2)\mathrm{D}_3\mathbf{k}. \qquad (3.4)$$

The latter, however, is preferably written in the form

$$\mathbf{D}=(\mathrm{D}_1\mathbf{i}+\mathrm{D}_2\mathbf{j}+\mathrm{D}_3\mathbf{k})-\beta^2\mathrm{D}_3\mathbf{k}=\mathbf{D}_0-\beta^2\mathrm{D}_2\mathbf{k}. \qquad (3.5)$$

It must therefore again be reiterated that there is nothing classically in (3.5) to show that in $\mathbf{D}_0$ the real or unmasked displacement density of ρ is in any way affected by the motional equations (1.4) and (2.2). Nevertheless it is the latter two motionals that have given rise to the modifying or rather masking term $\beta^2\mathrm{D}_3\mathbf{k}$ occurring in (3.5). The consequence of the latter term is therefore not so much to "concentrate" the charge ρ on the fixed body as it is to mask the aggregate D_3 component—presumably by virtue of a Maxwellian type of displacement measured by the motionals.

The aether displacement quality D_3 is weakened as it were by the motion.

To calculate the "apparent charge" on the fixed body the following integral will need to be considered :

$$q=\int_S \mathbf{D}\, d\mathbf{S}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.6)$$

Naturally when the medium comes to rest with $\mathbf{w}=0$ the original charge q_0 still obtains. As a matter of fact it is quite immaterial whether a rigid or non-rigid electron be considered in evaluating (3.6). The displacement over any envelope when properly taken will determine the amount of the effective divergence (or effective charge) existing within.

The significance of (3.6) is therefore that a charged body whilst in motion between deflecting condenser plates, as in the Kaufmann and Bucherer experiments, has its static or calculated q_0-deflexion become smaller than would otherwise be the case. At the same time the "current effect" due to a stream of electrons will also give rise to a smaller observable deflexion when an impressed magnetic field is set up.

4. If now the normally charged spherical body has three independent displacement components in three independent directions it will follow that, integrating over the spherical surface with respect to either D_3 on the one hand or D_1 or D_2 on the other, we have

$$\int_S D_3 . dS = \tfrac{1}{3} q_0. \quad . \quad . \quad . \quad . \quad . \quad (4{\cdot}1)$$

From this standpoint therefore with respect to the masking effect due to $\frac{u^2}{v^2}=\beta^2$ on an actual spherical electron the apparent charge during an idealized steady flight becomes

$$q=q_0(1-\tfrac{1}{3}\beta^2). \quad . \quad . \quad . \quad . \quad . \quad (4.2)$$

Clearly in such experiments as those of Kaufmann and Bucherer it is not, at least classically, justified to assume that q is constant and therefore that all variations of the observed ratio q/m are attributable to variations in the Newtonian mass m.

Turning to the Bucherer experiments as given in Richardson's 'Electron Theory of Matter,' p. 242, it becomes possible to find what the effect would be if the normal electron whether at rest or in motion happened to be or become an ellipsoid with its longer axis in the direction of motion. For the more general case the factor $\frac{1}{3}$ in equation (4.2) would be transformed into a factor K, which might or

might not be a function of the translational velocity. Classical theory has so far no way of foretelling the direct action of the aether on an electrically charged particle during flight. This has already been pointed out by Heaviside (see his 'Electromagnetic Theory,' i. p. 274). It will be found, however, that taking

$$K \equiv \cdot 58, \quad q = q_0(1 - K\beta^2) \quad . \quad . \quad . \quad (4.3)$$

gives rather a remarkable agreement with the calculations made on the basis of the Einsteinian formula

$$m = \frac{m_0}{\sqrt{1-\beta^2}}. \quad . \quad . \quad . \quad . \quad . \quad (4.4)$$

Thus the Lorentz formula for the Bucherer experiment is deduced by Richardson (*l. c.* p. 241), giving

$$\left(\frac{e}{m_0}\right) = \left[\frac{q_0}{m_0}\right] = \frac{2cz}{H(a^2+z^2)} \cdot \frac{\beta}{\sqrt{1-\beta^2}}. \quad . \quad (4.5)$$

However, employing equation (4.3) above we have on the basis here suggested in contradistinction that

$$\left(\frac{e}{m_0}\right)_{\text{New}} = \frac{2cz}{H(a^2+z^2)} \cdot \frac{\beta}{1-K\beta}. \quad . \quad . \quad (4.6)$$

This means therefore by division that

$$\left(\frac{e}{m_0}\right)_{\text{New}} = \frac{\sqrt{1-\beta^2}}{1-K\beta^2} \cdot \left(\frac{e}{m_0}\right)_{\text{Lorentz}},$$

$$\equiv \frac{\sqrt{1-\beta^2}}{1-\cdot 58\beta^2} \cdot \left(\frac{e}{m_0}\right)_{\text{Lorentz}}. \quad . \quad . \quad (4.7)$$

The following table shows the results to three significant figures of employing the relationship (4.7) :—

β.	$\left(\frac{e}{m_0}\right)_{\text{Lorentz}}$	$\left(\frac{e}{m_0}\right)_{\text{New}}$.
·3173	1·752	1·762
·3787	1·761	1·78
·4281	1·76	1·78
·5154	1·763	1·785
·6870	1·767	1·767

. . (4·8)

It needs to be remembered that after all radiational effects due to acceleration in the direction of the path of the electron are being neglected. To what extent this can influence the

difference between the spherically applicable quantity $\frac{1}{3}\beta^2$ as against the above ellipsoidally taken $\cdot 58\beta^2$ is not so easily discernible.

In Heaviside's 'Electromagnetic Theory,' iii. pp. 471, 475, the question of the influence of acceleration on the Kaufmann and Bucherer experiments was discussed. Quite independently of the classical derivation of the longitudinal mass m_1 (linear velocity u only) and transverse mass m_2 (directional change in u only) there was still to be considered the influence of radiational waste of energy on both m_1 and m_2 in the theory of action.

Jan. 1932.

LXXXIV. *The Internal Energy of Gases after Explosion.* *By* W T. David, J. R. Brown, *and* A. H. El Din *.

Introduction.

IT is well known that when inflammable gaseous mixtures of the same composition are exploded in closed vessels of similar type but of different sizes, the maximum pressures developed after explosion are greater in the larger vessels than in the smaller ones. Those who hold the view that combustion is complete in any mixture as soon as it is completely inflamed and that combustion is complete in closed vessel explosions by the time maximum pressure is reached suggest that the differences in the pressures attained in the various vessels are to be explained wholly in terms of greater relative heat loss in the smaller vessels than in the larger ones owing to the greater surface-volume ratio in the former vessels. Those who believe, on the other hand, that the combustion process takes an appreciable time to complete itself, and is in general incomplete at maximum pressure, have felt that these pressure differences are in a large measure due to combustion being more complete in the larger vessels.

In order to attempt to decide between these views we have made explosion experiments in two spherical vessels, one of 6 inches and the other of 17·45 inches internal diameter, recording with a high degree of

* Communicated by the Authors.

accuracy the pressures attained and at the same time measuring the heat loss up to the moment of maximum pressure. These experiments, which have extended over the last two years, have shown that the difference in the relative heat losses up to this moment in the two vessels when mixtures of the same composition are exploded in them is surprisingly small. Generally in fast-burning mixtures it is less (often far less) than 1 per cent. of the heat of combustion of the mixtures, while even in fairly slow-burning mixtures this difference is not very much greater. The maximum pressures, or mean gas temperatures inferred therefrom, developed in the two vessels differ by far more than can be accounted for in terms of the relative percentage heat-loss differences, and the conclusion seems irresistible that combustion is more complete at the time of their development in the larger vessel than in the smaller one.

Some luminosity and platinum thermometry experiments made in this laboratory * led tentatively to the suggestion that there are two stages in the combustion process even in the simple gases, viz., (1) the combination stage resulting in the formation of molecules of CO_2 and H_2O possessing latent energy, and (2) the stage during which these molecules pass into the normal molecule and in so doing yield their latent energy. The experiments described in the present paper do not, of course, throw any light upon this suggestion, but it is important to bear in mind that the term "combustion" as used in this paper refers to the overall process of conversion of chemical energy into thermal energy in which the molecules of the products of combustion (CO_2 and H_2O) have attained their normal condition.

It will be remembered that the explosion method is the one usually employed in determining the specific heat of gases at high temperatures. Our experiments seem to prove definitely that specific heat values determined by this method are, in general, serious over-estimates of the true values, and that the determination of true values will be impossible until means are found for obtaining information relating to the amount of incomplete combustion in individual explosions.

* David and Davies, Phil. Mag. ix. p. 390 (March 1930).

In this connexion we may say that we hope to be able to arrange experiments with still larger vessels, and also to make further investigations with the Hopkinson recording calorimeter * ; but while such experiments cannot, we think, fail to produce interesting results, cumulative experience obtained over a number of years in our laboratory makes us doubtful whether the explosion method can, in the present state of knowledge, yield true specific heat values.

Description of Apparatus and Experimental Procedure.

The two cast-steel explosion vessels employed were spherical in shape, one 6 inches and the other 17·45 inches in internal diameter.

The maximum pressure developed during an explosion was measured from a continuous pressure record taken by photographing the movement of a spot of light reflected from a diaphragm indicator of special design.

At first our experience of diaphragm indicators was confined to those employing steel diaphragms. The pressure curves obtained from certain explosions suggested that steel was quite useless for the purpose, and further experiments designed to test the value of steel in this connexion confirmed our belief. The need for a material possessing suitable elastic properties, and one which did not distort appreciably on being heated by the exploding gases, was satisfied in the choice of invar.

The use of invar necessitated special precautions in calibrating. An increase in the temperature of the sphere caused mechanical stretching of the diaphragm, and hence a change in calibration, for the steel body of the indicator, by means of which the diaphragm was held in position, dilated with a rise of temperature.

During the early stages of an explosion there is but little change in the temperature of an explosion vessel, but the temperature varies from one explosion to another. The two spheres were calibrated, using the dynamic method, at various initial temperatures, the calibration varying in accordance with the figures published by Thorp †, who assisted us in developing the indicator.

For the purpose of measuring heat loss a platinum bolometer was fitted in each vessel. The bolometers

* Proc. Roy. Soc. A, lxxix. p. 138 (1907).
† Phil. Mag. viii. p. 814 (Dec. 1929).

were circular in shape, cut in the form of a grid, and fixed to a bakelite backing *. The bolometer in the large vessel was 2 inches diameter and 0·08 inch thick, that in the small vessel $1\frac{1}{2}$ inches diameter, 0·06 inch thick. They were placed with their surfaces flush with the interior surfaces of the explosion vessels, and arrangements were made whereby they could be moved to various positions in the vessels. In every experiment with any given mixture strength each bolometer was placed in at least two positions, one near the top and the other near the bottom. The surface of the bolometers and the interior surface of the vessels were covered with a very thin layer of dull black paint †.

The method of recording the rise of resistance of the bolometer, and therefore of its rise of temperature, was very similar to that employed by Hopkinson ‡, and need not be described here; but it should be stated that the reflecting galvanometer used was of the torsion-string type, having a natural period of one-fiftieth of a second.

The galvanometer deflexion giving the rise of resistance of the bolometer and the indicator deflexion giving the variation of pressure within the vessel were recorded simultaoneously on the same photographic film.

The bolometers were calibrated on numerous occasions by immersing them in a Dewar flask containing distilled water, the temperature of which was measured by means of an accurate mercury thermometer. As the bolometers were mounted on their bakelite backings during calibration, an hour was allowed to elapse before the deflexion corresponding to any particular temperature was read, in order to ensure that the bolometers were at the temperature of the water.

The sensitivity of the galvanometers was measured by passing a known current through them. This was done immediately after an explosion, and the resulting deflexion was photographed on the film. The sensitivity proved to be nearly constant, the variation being not more than 2 per cent. throughout the whole series of experiments.

* For the excellent mounting of the platinum on the bakelite backing we are indebted to Messrs. A. Kershaw and Son, Leeds.

† The insulating effect and heat capacity of the thin layer of paint were found to be negligibly small.

‡ Proc. Roy. Soc. A, lxxix. p. 138 (1907).

In the early stages of the work many experiments were made to test the accuracy of the bolometric measurements. Thus, for example, tests were made with two bolometers of different thermal capacity to measure the heat loss from explosions of the same strength of mixture. These experiments gave the same percentage heat-loss results.

The amount of heat lost to the bakelite backing was estimated, using the method of Hopkinson *. For this purpose the specific heat and the thermal conductivity of bakelite were determined, the latter by using an apparatus designed by Drs. Speakman and Chamberlain †, to whom we express our indebtedness. The loss to the backing up to the moment of maximum pressure was found to be negligible in most of our experiments.

All the inflammable mixtures used were prepared in a tank of 12 cubic feet capacity from gases stored in high pressure bottles. The mixture strengths as given have been corrected for the small impurities present in the various bottles. Analyses were made in a Bone and Wheeler apparatus.

Both vessels were charged with an explosive mixture from the storage tank in exactly the same manner. Each was exhausted to a pressure of 5 millimetres of mercury absolute, filled with mixture twice, and then the charge to be fired was admitted.

Ignition, by means of an electric spark, took place centrally in each vessel.

Four explosions of each mixture strength were made in each vessel; two of these were made with the bolometer in the top position and two with it in the bottom position. The bolometer readings up to the instant of maximum pressure in the two positions were very nearly the same. The mean of the readings has been taken in assessing the heat loss to the entire surface.

Experimental Results.

Two inflammable gases, carbon monoxide and hydrogen, were used in the experiments.

Carbon Monoxide-Air Explosions.—These constitute a series of explosions of mixtures ranging from those

* Proc. Roy. Soc. A, lxxxiv. (1910).

† 'Journal of the Textile Institute,' xxi. (1930).

rich in carbon monoxide to those rich in oxygen. Each mixture contained 1 per cent. of hydrogen in order to speed-up combustion, and this is included in the percentage of carbon monoxide given.

The heat of combustion, of carbon monoxide in cals. per gm. mol. at 20° C. has been taken as 68000 ; that of hydrogen 57290 cals. per gm. mol.

The main results of experiments are given in Table I. The initial state of the gases was governed by atmospheric conditions, the initial temperature being of the order of 20° C. The symbols T and P are used to represent absolute temperature and absolute pressure respectively, while the initial and maximum mean conditions are indicated by the use of the suffixes i and m. Since the ratio of the maximum pressure to the initial pressure depends on the initial temperature, all the ratios have been adjusted to an initial temperature of 20° C. The corrected ratios are designated $\left[\frac{P_m}{P_i}\right]_{20°}$. The heat lost up to the instant of maximum pressure expressed as a percentage of the heat of combustion is referred to as h_m in the table, and the time of attainment of maximum pressure, measured in seconds from the passage of the spark, as t_ϵ.

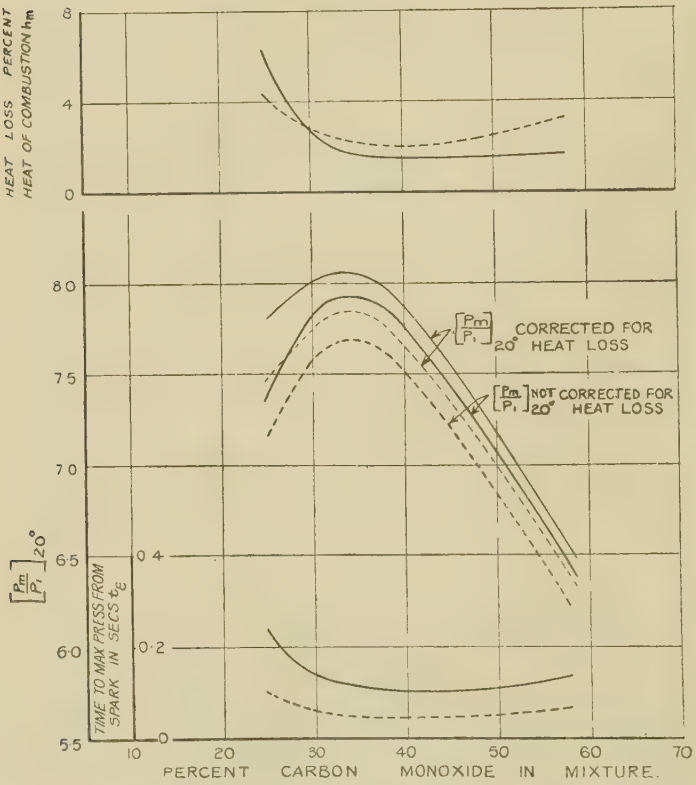
Curves showing the variation of $\left[\frac{P_m}{P_i}\right]_{20°}$, h_m, and t_ϵ, with the percentage of carbon monoxide, are given in fig. 1, those relating to the 6-inch vessel being shown dotted. The pressure ratios associated with the large vessel are greater than those with the small vessel. In general the percentage heat-loss differences in the two vessels are surprisingly small. In the case of the weakest mixture the heat loss in the large vessel is greater than that in the small sphere, but elsewhere there is a bigger percentage heat-loss in the small vessel.

In order to estimate the effect of the heat losses on the pressures developed the maximum mean rises of temperature have been calculated from the maximum pressures developed in the explosions. The energy held * by each mixture at the instant of maximum pressure has been estimated, using the specific heat

* In excess of that at 20° C.

values collected by Partington and Shilling *. Next it has been assumed than an amount of heat equal to that actually lost is given to the gases, and the resulting temperature-rise calculated. From these corrected temperature-rises pressure ratios corresponding

Fig. 1.



Curves showing actual $\left[\frac{P_m}{P_i}\right]_{20°}$, percentage heat loss and $\left[\frac{P_m}{P_i}\right]_{20°}$ corrected for heat-loss in explosions of carbon-monoxide and air. The full line curves relate to explosions in the 17·45-inch sphere and the dotted curves relate to explosions in the 6-inch sphere.

to them have been calculated. Errors in the specific heat data will have but little effect on the results.

* 'The Specific Heats of Gases,' Benn (1924).

The values of the pressure ratios corrected for heat loss in the manner indicated above are plotted in fig. 1, the actual values being given in Table I. If heat loss accounted for the actual pressure differences in the two vessels, then the pressure ratios, when corrected for heat loss, should be the same for each vessel. It will be seen that there are considerable differences, and thus the fact is established that heat loss plays but a small part in causing the actual differences of pressure which are found when mixtures are exploded in vessels of similar type but of different sizes.

Explosions of Hydrogen-Air Mixtures.—Hydrogen-air mixtures of different strengths were exploded in the two spheres from atmospheric conditions of pressure and temperature, and the data relating to these experiments are given in Table II.

Except in the case of the weakest mixture—that containing 12·37 per cent. H_2—no appreciable rise in the temperature of the bolometers could be detected up to the instant of maximum pressure. As will be seen from the table the heat loss in the small vessel was a little greater than that in the large vessel in the case of the 12·37 per cent. H_2 mixture.

The curves in fig. 2 show the pressure ratios $\left[\frac{P_m}{P_i}\right]_{20°}$, and the times of explosion t, plotted against the mixture strength, the dotted curves relating to the 6-inch sphere. The pressure ratios given by the large vessel are everywhere greater than those associated with the small one, and since heat loss with one exception is negligible, the difference can only be attributed to combustion being in a more advanced stage at the instant of maximum pressure in the large vessel. When correction is made for heat loss in the case of the weakest mixture combustion still appears to be more advanced in the large vessel, for the pressure ratios become 4·91 for the large vessel and 4·78 for the small one.

Discussion.

It has been shown that when mixtures of identical composition are exploded in similar vessels of different sizes higher pressures are developed in a large vessel than in a smaller one and that the higher pressures

TABLE I.

Mixture.	Large vessel.					Small vessel.				
	$\left[\frac{P_m}{P_1}\right]_{20^\circ}$.	h_m.	t_i.	$(T_m - T_1)$.	$\left[\frac{P_m}{P_1}\right]_{20^\circ}$ corrected for heat loss.	$\left[\frac{P_m}{P_1}\right]_{20^\circ}$.	h_m.	t_e.	$(T_m - T_1)$.	$\left[\frac{P_m}{P_1}\right]_{20^\circ}$ corrected for heat loss.
25 per cent. CO + air *	7·35	6·3	0·229	2170	7·8	7·15	4·25	0·094	2110	7·47
29·7 per cent. CO + air	7·8	2·9	0·132	2380	7·99	7·55	2·9	0·056	2290	7·73
34 per cent. CO + air.....	7·92	1·8	0·115	2400	8·05	7·69	2·4	0·045	2320	7·85
39·6 per cent. CO + air	7·8	1·6	0·099	2320	7·91	7·55	2·1	0·042	2240	7·69
47 per cent. CO + air.....	7·3	1·7	0·1	2110	7·41	7·07	2·3	0·046	2035	7·22
58 per cent. CO + air.....	6·45	1·6	0·132	1780	6·53	6·22	3·3	0·061	1710	6·4

* The air used was obtained from a bottle, and found to contain 20·6 per cent. of oxygen; 1 per cent. of hydrogen is included in the carbon-monoxide given.

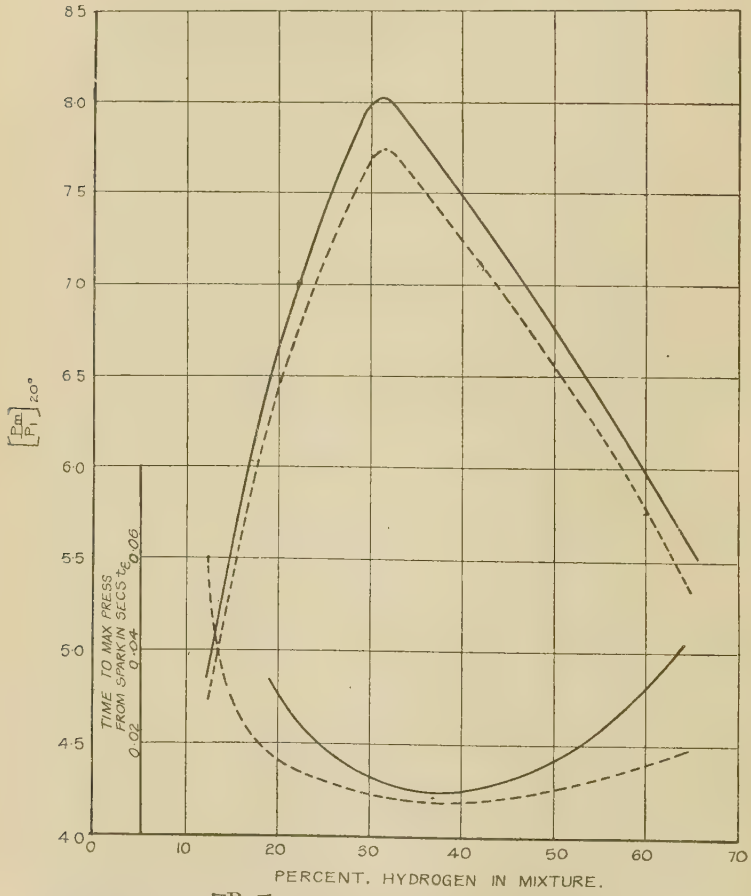
Table II.

Mixture.	Large vessel.				Small vessel.			
	$\left[\frac{P_m}{P_1}\right]_{20}°$	h_m.	t_6.	$(T_m - T_1)$.	$\left[\frac{P_m}{P_1}\right]_{20}°$	h_m.	t_6.	$(T_m - T_1)$.
12·37 per cent. H_2+air *	4·87	1·0	0·142	1230	4·72	1·5	0·06	1180
19·9 per cent. H_2+air	6·62	Very small, certainly less than 0·25 %.	0·032	1860	6·41	Very small, certainly less than 0·25 %.	0·016	1800
29·85 per cent. H_2+air	7·97		0·013	2440	7·66		0·01	2330
34·8 per cent. H_2+air	7·87		0·011	2360	7·58		0·008	2250
39·8 per cent. H_2+air	7·53		0·01	2200	7·27		0·008	2110
49·75 per cent. H_2+air	6·84		0·018	1940	6·62		0·01	1870
54·7 per cent. H_2+air	6·42		0·022	1770	6·31		0·013	1730
59·7 per cent. H_2+air	6·03		0·032	1630	5·82		0·016	1560
64·6 per cent. H_2+air	5·62		0·042	1470	5·37		0·019	1400

* The air used was obtained from a bottle and found to contain 19·9 per cent. of oxygen.

cannot be explained to any considerable extent in terms of lower relative heat loss. It seems clear therefore that combustion is more complete at the moment of maximum pressure in the large vessel than in the smaller

Fig. 2.



Curves showing $\left[\frac{P_m}{P_i}\right]_{20°}$ and time of explosion in explosions of hydrogen and air. The full line curves relate to explosions in the 17·45 inch sphere and the dotted curves relate to explosions in the 6-inch sphere.

one, and if this is so, it is hardly likely that combustion is complete even in the large vessel *.

* Inference from experiments with a sphere of intermediate size—a 12-inch sphere—seems to confirm this.

Evidence has been accumulating in our laboratory over a long period of years which offered strong support to the incomplete combustion theory, but much of this has rested upon certain assumptions, such as, for example, that the specific heats of all diatomic gases used as diluents in explosive mixtures were practically the same over any given temperature range *. The recent work of Henry † upon the specific heats of oxygen and nitrogen seems, however, to throw some doubt upon this assumption; but it seems to us that our present experiments offer conclusive proof of the incomplete combustion theory in that it is independent of all such assumptions.

The explosion method as usually employed cannot therefore be relied upon to furnish accurate information in regard to the specific heat of gases at high temperatures—even when measured heat losses are taken into account—unless and until means are provided by which the amount of incomplete combustion can be estimated.

In order to make this clear we give in Table III. the specific heats of the triatomic products of combustion calculated from a few of our experiments in the usual manner. In view of the fact that we believe we have demonstrated the correctness of the incomplete combustion theory we call these calculated specific heats the "apparent" specific heats. In our calculations we have taken the specific heat values for the diatomic diluents from the tables of Partington and Shilling, but it should be pointed out that any error in these values affects only to a very small extent the relative values of the triatomic apparent specific heats calculated for the two vessels, in that for any given mixture the diatomic diluents are the same and the temperature ranges very nearly the same in each case. The experiments selected relate to over-rich mixtures, and temperature dissociation is therefore small.

It will be noted that for any given mixture strength the apparent triatomic specific heats are considerably greater when determined from the small vessel data than when determined from the large vessel data, in spite of the fact that the temperature range is somewhat larger in the large vessel. This, it will be clear, we

* Thorp, Phil. Mag. viii. p. 814 (1929).
† Proc. Roy. Soc. A, cxxxiii. p. 492 (Sept. 1931).

Table III.

Inflammable mixture. Initial pressure-atmospheric.		Maximum pressure after explosion (lb. per sq. in. above atmos.).	Approx. maximum mean temperature (° C.).	Measured heat loss per cent. heat of combustion.	Mean specific heats of triatomic products of combustion—20° C. to explosion temperature.	
					From explosions.	Partington and Shilling.
(40 per cent. CO+1 per cent. H_2) and air	6″ sphere..	95·2	2220	2·2	15·55	about 10·05 *
(Air from bottle containing 20·6 per cent. O_2)	18″ „ ..	98·7	2300	1·6	14·4	about 10·15 *
(50·7 per cent. CO+1 per cent. H_2) and air	6″ „ ..	84·2	1920	2·6	15·4	9·75
(Air from bottle)	18″ „ ..	87·6	1990	1·6	14·4	9·81
40 per cent. H_2 and air	6″ „ ..	96·7	2230	Very small, certainly less than ¼ per cent. of the heat of combustion.	12·0	9·1
(Atmospheric air)	18″ „ ..	100·5	2330		10·65	9·32
52·5 per cent H_2 and air	6″ „ ..	86·3	1920		10·7	8·4
(Atmospheric air)	18″ „ ..	89·0	1985		9·5	8·54

* Extrapolated.

attribute to more complete combustion in the large vessel.

In the last column of the table we give the Partington and Shilling values appropriate to the temperature ranges. These are in all probability nearer to the true values than those determined by the explosion experiments. It will be seen that in the case of CO_2 the P. and S. values are very much lower than the explosion values, but in the case of H_2O the difference is not so great. We conclude from this that while combustion is more complete in the large vessel than in the smaller one it is seriously incomplete in both vessels in the CO explosions, but much less so in the hydrogen explosions.

We wish to thank the Department of Scientific and Industrial Research for a maintenance grant for one of us (J. R. B.), which enabled us to do certain preliminary work bearing upon the present experiments, and also the Egyptian Government for a maintenance grant for another of us (A. H. El D.).

LXXXV. *The Experimental Demonstration of the Effects of Coupling of Harmonic Motions, as illustrating Phenomena in Quantum Mechanics.* *By* A. C. Burton, *Ph.D.* *†

IN textbooks on the new quantum mechanics, when the phenomenon that is variously called "equivalence degeneracy," the "exchange phenomenon," or "quantum resonance" is introduced, it is usually pointed out that the classical problem of coupled pendulums has certain features that are mathematically analogous. The case of coupled electrical circuits, however, which is the electrical analogue of that mechanical problem, is rarely mentioned. Yet for the purpose of lecture demonstration the electrical case, while mathematically equivalent, offers many advantages over the mechanical. Accordingly the apparatus to be described was constructed and proved successful in illustrating in a spectacular manner the principles under discussion.

* Communicated by Professor J. C. McLennan, F.R.S.

† Holder of a Fellowship under the National Research Council of Canada.

It is appropriate to give here a brief sketch of these principles, such as might be given by one demonstrating the experiments. The simplest atoms dealt with in the quantum theory are those which resemble hydrogen in that for the prediction of their spectra a single electron only has to be considered with its characteristic set of quantum numbers. Here the new mechanics, as the old, predicts a single resulting set of discrete energy levels or "terms," which, if "spin" is not considered, are all "singlets." When we pass to helium-like atoms there are two electrons to be considered which in the normal state of the atom are entirely "equivalent" to each other. A new phenomenon then becomes evident that was quite unexpected on the old quantum theory, the explanation of which on the new quantum mechanics was first given by Heisenberg * and by Dirac † independently. Instead of the single set of terms similar to those of hydrogen there are now, theoretically predicted and experimentally observed, two quite distinct sets of terms. For a given set of quantum numbers characterizing the system there are two distinct solutions of the equations, giving two energies that differ by the "exchange interaction energy." One of these solutions is characterized by being symmetrical in the wave functions; the other is antisymmetrical. When "spin" is introduced the requirements of resultant symmetry demand that in one system the spins should be anti-parallel, giving "singlet" terms, in the other parallel, giving "triplet" terms. So distinct and non-combining are the two systems of terms that they were originally attributed to two kinds of helium, "orthohelium" and "parhelium." The two electrons may be pictured as exchanging places in their orbits periodically without radiation, the probability of finding one of them in either orbit being the same. The name "quantum resonance" was given because of the formal similarity between the equations in quantum mechanics and those describing two simple harmonic oscillators that are loosely coupled.

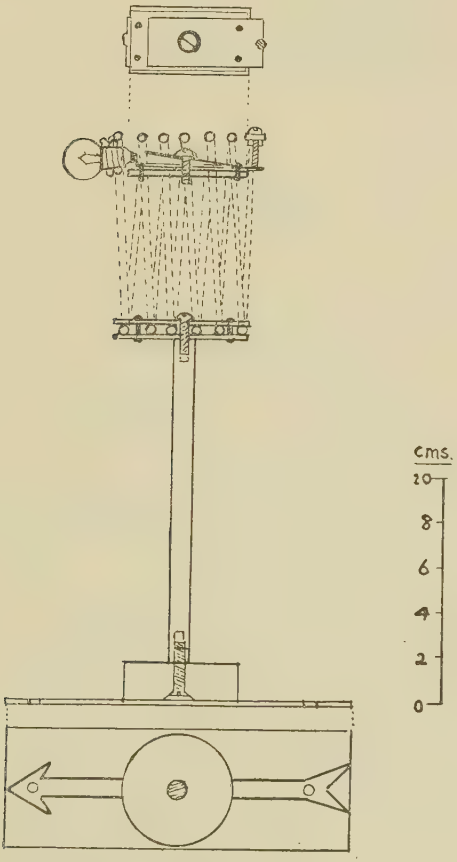
To illustrate the mechanical analogue two identical simple pendulums are "coupled" by connecting in some way their points of support. This may conveniently be done by suspending them from a common horizontal string whose tension and length determine the amount of coupling. If one pendulum be now given an amplitude of oscillation,

* Heisenberg, *Zeit. f. Phys.* xxxviii. p. 411 (1926); xxxix. p. 499 (1926); xli. p. 239 (1926).

† Dirac, Proc. Roy. Soc. A, cxii. p. 661 (1926).

the other being initially at rest, the energy of oscillation is gradually exchanged between the two until the first has been reduced to rest and the second has reached its maximum amplitude, when the exchange of energy proceeds in the reverse direction. The details of the progressive changes

Fig. 1.



Resonator.

in amplitude and in phase are fully described in textbooks on dynamics. In the textbooks of radio engineering also, such as Morecroft's 'Principles of Radio Communication,' p. 264 (second edition), the problem of coupled pendulums is treated as offering an analogue to coupled electrical circuits. The phenomenon of "exchange" of energy is

plainly demonstrated by the mechanical experiment, but the fact that there are two time periods associated with the system instead of the single period proper to each pendulum alone is not immediately obvious.

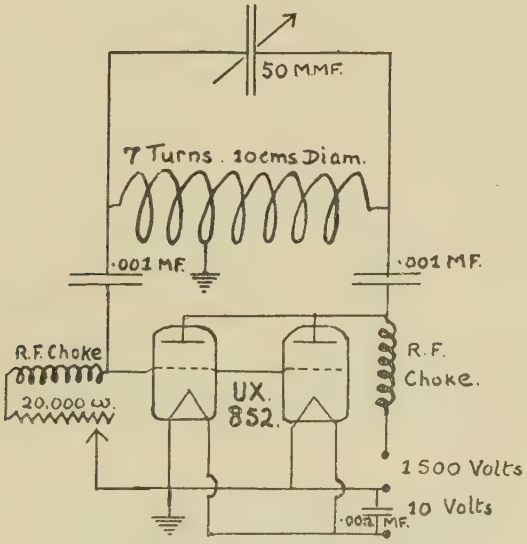
For the electrical demonstration which it is the purpose of this paper to present the resonant circuits are as illustrated in the scale-drawing (fig. 1). In each an inductance, an adjustable capacity, and a resistance, the latter being the filament of a "flashlight" bulb, are connected in series. The inductance is of aluminium rod, bent into a helix of five turns of diameter 8·5 cm. Aluminium, while light and a good conductor, is stiff enough to require no supporting material that would introduce undesirable stray capacity between the turns. At one end the rod is bent into a small spiral into which a miniature flashlight bulb screws tightly. The resistance of the latter should be as low as possible, those bulbs rated at 1·25 volts and having a resistance of about 2 ohms proving excellent in operation. The condensers are constructed after the pattern of "trimmer" condensers sold by manufacturers of radio parts. By adjustment of the centre screw their capacity can be varied from about 30 to 50 mmf. The condenser is soldered on the one side to the screw clamped to one end of the inductance, on the other to the protruding electrode of the flash-lamp bulb. The mounting of the resonator to an ebonite rod and brass base is conveniently made by two fibre strips rivetted in position and by a single screw. Three such resonators were constructed of identical dimensions.

The generator is designed to give waves of from 25 to 30 metres wave-length, and employs the familiar Hartley circuit, details of which are shown in the diagram (fig. 2). The frequency is regulated by the tuning condenser of 50 mmf. capacity connected across the oscillation coil. Its long aluminium pointer moves over a white card, on which a scale of frequency may be marked. Vernier motion from a distance is secured by an ebonite rod 18 inches long, whose rubber-covered end in rotating bears against the edge of the dial of the condenser. The two UX 852 valves in parallel give an output of up to 125 watts. Any design of oscillation generator giving waves of frequency of the desired range and of sufficient intensity is of course equally suitable.

When the resonator is placed in the field and the frequency of the generator is continually altered by rotation of the condenser dial the lighting up of its lamp indicates resonance at a sharply defined point on the frequency scale.

If, with the generator giving waves of this resonant frequency, the resonator is now rotated slowly about its ebonite support as axis, the direction of the maximum field intensity is indicated by the maximum brightness of the lamp. On the base of one of the resonators is fixed a brass plate bearing a painted arrow which is parallel to the axis of the inductance mounted above. Two holes at the ends of the arrow, through which pencil marks may be made, enable the lines of field intensity to be plotted by the usual procedure. Slight detuning of the generator prevents

Fig. 2.



Circuit of generator 25–30 metres.

burning out of the lamp when the resonator is close to the generator.

By careful adjustment of the "trimmer" condensers of the resonant circuits the three were made to have, when each is alone in the field, a frequency of resonance at the same point of the scale, which for convenience was made the centre point. They were then arranged in the field, one being isolated the other two loosely coupled to each other. The effect of this coupling is now plainly seen as the frequency of the generator is varied from one side of the scale to the other. At a frequency displaced to the long-

wave side of the original point the lamps of both of the coupled resonators light up together. On further increase of frequency the lamps go out, and at the original frequency the isolated resonator indicates that its resonant point is unaltered. At a point displaced to the short-wave side the two coupled circuits once more light up simultaneously. A graphic picture is given of a two-electron atom absorbing radiation of two wave-lengths instead of the single wave-length proper to each electron alone. This "splitting" of the point of resonance can be noted when the "coupling" is as little as 1 per cent., which is obtained when the coils are coaxial and about 20 cm. apart. As they are brought close together and the coupling is increased the separation of the two resonance points increases, until when the coils are almost in contact the coupling reaches about 15 per cent., and a separation in frequency of about the same amount results between the resonances. The displacement of frequency is asymmetrical, being greater to the short-wave side.

The theoretical treatment of coupled circuits has been dealt with in great detail by many authors from Drude down, a list of whom is given by Chaffee *. In most cases a driving e.m.f. is supposed in one of the circuits, called the "primary circuit," whilst the second circuit is the "secondary." In our case, however, the two circuits are exactly similar, each having an e.m.f. induced in it by its coupling to the generator. Reduced to their simplest terms the differential equations for the currents I_1 and I_2 in the two circuits are

$$\left.\begin{aligned} M\frac{\partial^2 I_2}{\partial t^2}+L\frac{\partial^2 I_1}{\partial t^2}+R\frac{\partial I_1}{\partial t}+\frac{I_1}{C}&=\mu_1\frac{\partial^2 I}{\partial t^2}\\ M\frac{\partial^2 I_1}{\partial t^2}+L\frac{\partial^2 I_2}{\partial t^2}+R\frac{\partial I_2}{\partial t}+\frac{I_2}{C}&=\mu_2\frac{\partial^2 I}{\partial t^2}\end{aligned}\right\}, \quad . \quad (1,$$

where M, L, R, C have their usual significance of mutual inductance, self-inductance, resistance, and capacity. μ_1 and μ_2 are the mutual inductances between each coil and the generator coil which carries the current I. Assuming that the latter is sinusoidal, we put $I=I_0e^{j\omega t}$ and find solutions of the form

$$\left.\begin{aligned} I_1&=Ae^{j\omega t},\\ I_2&=Be^{j\omega t},\end{aligned}\right\}.$$

* Chaffee, Proc. Inst. Radio Engineers, iv. p. 3 (June 1916).

Substituting we find

$$\left.\begin{aligned} -BM\omega^2+AZ&=-\mu_1\omega^2I_0\\ -AM\omega^2+BZ&=-\mu_2\omega^2I_0\end{aligned}\right\}\ \text{writing Z for}\ \left(L\omega^2+Rj\omega+\frac{1}{C}\right). \quad . \ . \ . \quad (2)$$

Solving for A we find

$$A=-\frac{\omega^2I_0(\mu_1Z+\mu_2M\omega^2)}{Z^2-M^2\omega^4}.$$

The points of resonance are given by the values of ω that make the amplitude A of the current a maximum. For this we have the two conditions

$$\left.\begin{aligned} Z-M\omega^2&=0,\\ Z+M\omega^2&=0.\end{aligned}\right\}$$

Introducing now the coupling coefficient $k=\frac{M}{L}$ and the resonant frequency of the isolated circuits, $\omega_0=\frac{1}{\sqrt{LC}}$, these equations become if the resistance be neglected

$$(\omega_0{}^2-\omega^2)\pm k\omega^2=0,$$

or, the two frequencies that give maximum currents in the circuits are

$$\left.\begin{aligned} \omega'&=\frac{\omega_0}{\sqrt{1+k}}\ldots\ \text{the "long-wave" resonance,}\\ \omega''&=\frac{\omega_0}{\sqrt{1-k}}\ldots\ \text{the "short-wave" resonance.}\end{aligned}\right\}. \quad (3)$$

The displacement of the frequency of resonance does not then take place symmetrically as the coupling "k" is increased, the short-wave resonance suffering the greater displacement.

Eliminating I_0 from the equations (2) we get

$$\frac{A}{B}=\frac{\mu_1Z+\mu_2M\omega^2}{\mu_2Z+\mu_1M\omega^2}.$$

Inserting the value of Z, this becomes

$$\frac{A}{B}=\frac{\mu_1(\omega_0{}^2-\omega^2)+\mu_2k\omega^2+j\omega\mu_1\,R/L}{\mu_2(\omega_0{}^2-\omega^2)+\mu_1k\omega^2+j\omega\mu_2\,R/L}.$$

For the long-wave resonance we have

$$\omega_0{}^2-\omega^2=k\omega^2,$$

giving

$$\frac{A}{B}=\frac{k\omega^2(\mu_1+\mu_2)+j\omega\mu_1 R/L}{k\omega^2(\mu_1+\mu_2)+j\omega\mu_2 R/L},$$

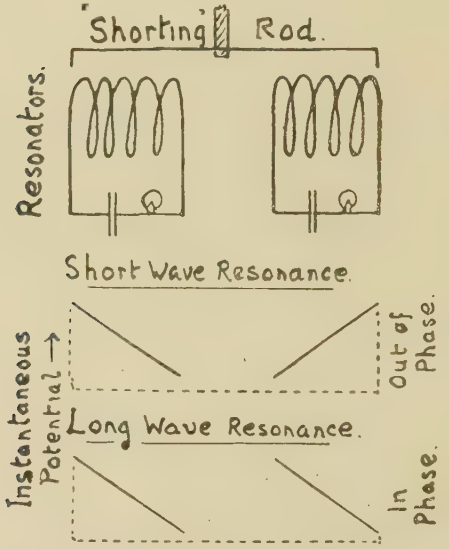
while for the short-wave resonance

$$\omega_0^2-\omega^2=-k\omega^2,$$

giving

$$\frac{A}{B}=\frac{-k\omega^2(\mu_1-\mu_2)+j\omega\mu_1 R/L}{k\omega^2(\mu_1-\mu_2)+j\omega\mu_2 R/L}.$$

Fig. 3.



It is seen that if R/L is small compared with ωk, *i. e.*, if R is small compared with ωM, at the long-wave resonance A and B have the same sign (A = B), while at the short-wave resonance they have opposite signs (A = −B). This means that in the former case the two currents are in phase, in the latter they are completely (if resistance be neglected) opposite in phase.

This fundamental difference between the two resonances, which corresponds to the conditions of "symmetry" and "antisymmetry" in the two-wave functions of the helium atom, can readily be demonstrated with the apparatus. The two resonators are placed in close coupling with their coils coaxial, and the generator is adjusted to be at the

point of long-wave resonance. A short bar of copper held on a long ebonite rod is then held over the coils so that its ends may be made to "short-circuit" the outer ends of the coils. When contact is made the lamps are completely extinguished. The currents are in phase and the two points connected are of different instantaneous potential (see fig. 3). The experiment is then repeated, but with the generator set at the short-wave resonance. In this case a slight dimming of the lamps may be seen, but the currents are comparatively unaffected. Here the currents are almost completely out of phase (completely only if the resistance were zero), and the points connected by the copper rod are therefore very nearly at the same instantaneous potential (fig. 3).

In practice it is found that the individual couplings of the two resonators to the generator must bear a certain relationship to each other in order that the magnitudes of the currents induced at the two points of resonance shall not be very different. It may be arranged that either the long-wave or the short-wave resonance will give the greater intensity in the lamps. Examination of the equation for the amplitude A shows that at these points of resonance the amplitude is approximately given by

$$A = \frac{I_0}{R^2}[\mu_1 Z + \mu_2 M\omega^2].$$

If now we have the special case of $\mu_1 = \mu_2$, then at the short-wave resonance, where $Z + M\omega^2 = 0$, the amplitude is zero. In this case only the long-wave resonance will exist. If, however, $\mu_1 = -\mu_2$, the long-wave resonance will be suppressed, since at that point $Z - M\omega^2 = 0$. The condition for equal amplitudes of the induced currents at the two resonance points is that $\mu_1 = k\mu_2$.

We may trace a connexion between this and the fact that in the lowest state of helium only the singlet state exists. In consequence when light is passed through normal helium the absorption that occurs is that connected with the singlet state, not with the triplet. The Pauli exclusion principle asserts that no state can exist in which the quantum numbers of two electrons are completely equivalent. To obtain the normal triplet level this would have to be the case, the spin quantum numbers of the two electrons being the same in addition to their other quantum numbers. We may associate this with the equality of μ_1 and μ_2 in the electrical case. For the higher, excited states the triplet terms are not excluded, as the other quantum numbers may here be

different, corresponding to the electrical case of μ_1 and μ_2 different. The case $\mu_1 = -\mu_2$ does not appear to correspond to any case in atomic theory, for here the normal singlet states would be excluded instead of the triplet states.

It can be shown mathematically that, as in the case of the coupled pendulums, energy is being interchanged periodically between the two circuits, but here the interchange is much too rapid for easy demonstration. Interesting and curious results are obtained if three identical resonators are mutually coupled. A large number of points of resonance result, the distribution of currents in which depends on the various mutual couplings and upon the individual couplings of the coils with the generator. With the two coils an interesting effect is obtainable when the coupling is made very close indeed. The short-wave resonance may then be observed to be itself split into two very close resonances. This "fine structure" is due, it is thought, to the introduction when the resonators are in close proximity of capacitative coupling in addition to the magnetic interaction.

Another phenomenon in quantum mechanics that may be illustrated by this apparatus is that of "perturbation" in band spectra, predicted by the theory of Kronig * and shown experimentally by Dieke † in the band systems of helium. When two molecular terms, fulfilling certain conditions, lie close together their mutual interaction has the effect that they seem to repel each other. To illustrate this the resonance frequency of one of the circuits is slightly altered by adjustment of its trimmer condenser, so that there are now two resonance points on the frequency scale, one belonging to each of the two circuits. When the latter are coupled it is found that the resonance points are further apart on the scale than before, as though there had been repulsion between them. Mathematical treatment similar to that already sketched, but for two "detuned" circuits whose natural frequencies are ω_1 and ω_2, gives for the two frequencies of resonance when the circuits are coupled

$$\left.\begin{aligned}(\omega_1')^2 &= \frac{\omega_1^2+\omega_2^2+\sqrt{(\omega_1^2+\omega_2^2)^2-4\omega_1^2\omega_2^2(1-k^2)}}{2(1-k^2)}\\ (\omega_2')^2 &= \frac{\omega_1^2+\omega_2^2-\sqrt{(\omega_1^2+\omega_2^2)^2-4\omega_1^2\omega_2^2(1-k^2)}}{2(1-k^2)}\end{aligned}\right\}; \qquad (4)$$

* Kronig. *Zeit. f. Phys.* l. p. 347 (1928).

† Dieke, 'Nature,' March 23, 1929, p. 446. The similar effect in atomic spectra has been discussed recently by Shenstone and Russell, Phys. Rev. xxxix. p. 413 (Feb. 1st, 1932).

whence we find

$$\left.\begin{aligned}(\omega_1')^2+(\omega_2')^2&=\frac{\omega_1^2+\omega_2^2}{(1-k^2)}\\ \omega_1'\omega_2^1&=\omega_1\omega_2,\end{aligned}\right\}.$$

These give $$(\omega_1'-\omega_2')^2=\frac{(\omega_1-\omega_2)^2+2\omega_1\omega_2k^2}{(1-k^2)}. \quad . \quad . \quad (5)$$

Equation (5) shows that the separation $(\omega_1'-\omega_2')$ is now greater than the initial separation $(\omega_1-\omega_2)$ of the resonance frequencies of the uncoupled circuits, and the perturbing effect increases as the initial difference $(\omega_1-\omega_2)$ is made less. In the spectral case the apparent repulsion is greater the more nearly coincident are the unperturbed energy levels.

The apparatus offers many other possibilities of useful demonstration. The effect of neighbouring conductors upon a circuit carrying alternating currents, known in general as the "shielding effect," is easily demonstrated. The effect of a neighbouring circuit, which may be considered to have resistance R and reactance X, upon a circuit to which it is coupled with a mutual inductance M is twofold. Let Z be the impedance of the shielding circuit, *i.e.*, $Z^2=X^2+R^2$. Then the reactance of the "shielded" circuit is effectively decreased by an amount $\left(\frac{\omega M}{Z}\right)^2 X$, while its resistance is effectively increased by $\left(\frac{\omega M}{Z}\right)^2 R$. A holder of non-conducting material is mounted upon an ebonite stand, into which plates of metal, about 1/16 inch thick and 6 inches square, can be fitted and interchanged. When the plate is brought near to one of the resonant circuits the twofold effect is plainly seen, first in the shift of the point of resonance to a higher frequency, and secondly in the reduced brightness of the lamp at the new point of resonance. It can be demonstrated that at these high frequencies a copper plate is just as effective a shield as an iron plate of the same thickness, while a plate of monel metal whose specific resistance is greater is noticeably less effective.

For the lecturer on alternating current theory or on radio engineering the apparatus which has been described serves to give graphic illustration of many points of theoretical and of practical importance. Every student, for instance, who is to use a wave-meter with accuracy should be impressed with the fact of the occurrence of "double resonance" in closely coupled circuits. For the

student of atomic theory it may, perhaps, give nothing but a measure of palatability to those mathematical abstractions of modern quantum theory that to the mental digestion trained on models are so unsatisfying.

McLennan Laboratory,
Physics Department,
University of Toronto.
July 25th, 1932.

LXXXVI. *The Cold Cathode Vacuum Arc.* *By* F. H. Newman, *D.Sc., A.R.C.S., Professor of Physics, University College, Exeter* *.

[Plate XX.]

1. *Introduction.*

THE current in many arcs appears to be carried at the cathode primarily by electrons, emitted thermionically. To neutralize the electron space-charge and to permit large currents to pass with low voltage some positive ions must exist throughout the arc, even in the neighbourhood of the cathode. The mobility of the positive ion is small compared to that of the electron, so that only a relatively small positive ion current is required to provide the positive space-charge. In the main the current at the cathode is electronic in nature, and to replenish the positive ions, the electrons emitted from the cathode, acquiring sufficient energy in passing through the cathode fall of potential, ionize the molecules of active gas, or vapour. Such is the simple theory for normal arcs having cathodes heated by means of an external source, or cathodes of refractory materials which reach a high temperature. Thus in both the normal tungsten and carbon arcs the cathode is at such a high temperature that the thermionic current density is equal to the measured current density.

With other types of arcs—those in which the temperature of the cathode does not exceed a few hundred degrees—this theory of thermionic emission is inadequate. Such arcs include those in which the arc and its cathode elec-

* Communicated by the Author.

trode are moved relative to each other with such velocity that the cathode never reaches a high temperature—the so-called " Stolt arc " *. The author † has also described an arc between refractory metals in which the cathode remains comparatively cold, so that there is no trace spectroscopically of the vapour from the metallic electrode in the arc's radiation. Again, with metals having high vapour pressures at moderate temperatures, such as cadmium, mercury, sodium, etc., it is inconceivable that the temperature of the cathode rises above a few hundred degrees, and so in many such cold cathode arcs there is no appreciable thermionic emission. Thus the current near the cathode cannot be carried by this process.

It has been suggested that in the case of an arc in which the cathode is not a heated filament the free electrons are thermions emitted by the cathode hot spot, and Stark's ‡ results seemed to suggest that with mercury the temperature of this spot is about 2000° K. This theory has been thoroughly disproved by the fact that the evaporation of mercury is far too slow to justify an assumption of temperature requisite for thermionic emission. Compton § has shown theoretically that the surface temperature of this cathode spot cannot exceed about 200° C.

Langmuir ‖ first suggested that in the mercury arc electrons may still constitute the major portion of the current at the cathode, but that instead of being emitted thermionically the electrons are pulled out of the relatively cold cathode by a very high field. Later this theory was modified by Compton ¶ and Mackeown **. Lamar and Compton †† have measured the cathode fall in the mercury arc, and they estimate it to be very nearly 10 volts, and although the actual distance across which this fall of potential occurs has never been measured, these experimenters estimate it to be less than $1{\cdot}76\times10^{-4}$ cm. Given a cathode fall of 10 volts in a distance less than $1{\cdot}76\times10^{-4}$ cm., the average field

* *Zeits. f. Phys.* xxvi. p. 95 (1924).
† Phil Mag. xi. p. 796 (1926).
‡ *Ann. d. Phys.* xii. p. 692 (1903).
§ Phys. Rev. xxxvii. p. 1077 (1931).
‖ Gen. Elec. Rev. xxvi. p. 735 (1923).
¶ Trans. A. I. E. E. xlvi. p. 868 (1927).
** Phys. Rev. xxxiv. p. 611 (1929).
†† *Ibid.* xxxvii. p. 1069 (1931).

in the fall space is greater than $5{\cdot}7 \times 10^4$ volts per cm. By means of the simple space-charge equation it is shown that the field at the cathode surface is 4/3 times the mean field, or greater than $7{\cdot}6 \times 10^4$ volts per cm. Of course the cathode surface is highly agitated and consequently rough. Thus the fields to points may greatly exceed the values calculated above, and may very likely attain values of the order of millions of volts per cm. Langmuir pointed out that fields of this magnitude are probably adequate to account for the observed electron emission from the cathode spot as "autoelectronic emission" or "field currents."

Slepian and Haverstick* criticise the strong field theory. They point out an important consequence of this view is that cold cathode arcs should not be able to exist with less than a minimum cathode current density of considerable magnitude. Thus, to draw electrons from a cold cathode surface requires gradients in excess of 10^6 volts per cm., so that if the molecular weight of the ion is less than 40 and the cathode fall of potential less than 20 volts, it should not be possible to have an arc with less than 1000 amperes per square centimetre. These experimenters, however, using different metals as cathode surfaces, found that the cathode current densities were all very decidedly less than this value. Their currents were of the order of 25 amperes, and the current density varied between 7 and 125 amperes per square centimetre. It seems to the author, however, difficult to estimate the exact size of the cathode spot. The latter may be extremely minute and its apparent size greatly overestimated due to the brilliant luminosity in its immediate vicinity. In connexion with this point evidence deduced by Stern, Gossling, and Fowler † from cold cathode currents drawn from small wires, seems to show that the current densities are extremely high, though the emitting areas are very small. Whether the same conditions result in the case of the arc is, of course, uncertain; even if they do, the emitting spots may be so small and so scattered about that a very large gross positive ion current will be needed to maintain a high average field over a large area.

* *Ibid.* xxxiii. p. 52 (1929).
† Proc. Roy. Soc. A, cxxiv. p. 699 (1929).

Slepian and Haverstick suggest that the high cathode gradient theory of the arc is inapplicable to arcs at moderately low pressures because of their estimated low current density. Slepian * had previously pointed out that at atmospheric pressure, if the layer of gas next to the cathode is sufficiently highly ionized, all the arc current can be carried to the cathode by the positive ions, so that an electron current from the cathode is unnecessary. These experimenters also found that the cathode current density decreases with pressure, and from this they infer that it is still possible for positive ions to carry the whole current. If the cathode current density remains high, as at atmospheric pressures, more than 100 per cent. ionization would be necessary to supply enough positive ions to carry the current. The suggestion was also made that the gas layer next to the cathode might be kept highly ionized by being maintained at a very high temperature, and calculations showed that the electrical energy input into this layer was more than enough to keep it hot. Similar calculations made with these low pressure, low current density arcs lead to entirely similar conclusions. The theory, however, fails to suggest any physical mechanism for the input of energy into this assumed high temperature region.

A difficulty in any theory depending primarily on electron emission from the cathode has been found in attempting to calculate, from the heat balance at the cathode, the fraction of the current carried by electrons, since such attempts have led consistently to values of this fraction which are too small to reconcile with the necessary amount of ionization, in view of the small cathode fall. For example, Compton † has shown that substituting direct experimental values of observable quantities in the heat balance equation for the mercury arc has led to the conclusion that about 50 per cent. of the current at the cathode is carried by electrons. Certain reasonable modifications have raised this to about 70 per cent. By straining every possible factor this proportion has been raised to about 87 per cent. These values are all too small to reconcile with the fact that

* Phys. Rev. xxvii. p. 407 (1926).
† Phys. Rev. xxxvii. p. 1077 (1931).

the cathode fall in the mercury arc is only about 10 volts, for 50 per cent. current carried by electrons would require 100 per cent. efficiency of ionization by them in order to obtain an equal number of positive ions to carry the remaining half of the current at the cathode. Even 87 per cent. would require at least 12 per cent. efficiency of ionization, which is still far in excess of any observed efficiencies of ionization by impact of electrons whose energy is near the minimum ionizing energy, 10·4 volts.

In calculating the heat balance at a cold cathode there are many factors to be considered. First there is the heating by impacting positive ions. These acquire energy in passing through the cathode potential fall, and since the thickness of the fall space is considered to be less than $1{\cdot}76\times10^{-4}$ cm., and since the ionic mean free path is considerably greater than this, collisions between ions and vapour molecules in this space may be neglected. The ions have negligibly small kinetic energy just before they enter the space, but not all of their kinetic energy is delivered to the cathode. Some of them will rebound from the cathode surface after neutralization thereat.

In addition, there is cooling by electron emission, cooling by radiation, cooling by evaporation, by gas conduction and convection, and by conduction through the cathode. There is also heating by energy derived from electrons in the cathode fall space and indirectly returned to the cathode. Compton, expressing the condition of thermal equilibrium by setting the total net rate of generation of heat at the cathode equal to zero, has shown that it may be made consistent with Langmuir's theory by making reasonable assumptions.

Whatever the mechanism of freeing electrons from the cathode, a certain minimum positive ion current is necessary to neutralize the negative space-charge. For this purpose, as many positive ions are needed if the electrons are pulled out under a high field as if they are emitted thermionically. The actual magnitude of the neutralizing positive ion current need not be very great—less than a thousandth part of the electron current for most ions. So if the electron current density is a few thousand amperes per square centimetre, a positive ion current of only a few amperes per square centimetre is sufficient.

2. *Experimental.*

Whatever the source of electrons in the cold cathode arc it is now generally agreed that they are not emitted thermionically from the cathode, and experiments by the author confirm this. The electrodes used in the present work were of pure tungsten, placed about 5 mm. apart in a discharge-tube. With a potential difference applied between them it was anticipated that an arc through the gas would be started by passing a momentary electrical discharge between one of these arc electrodes and a third one, the gas pressure being low—in the neighbourhood of 10^{-2} mm. of mercury. Although this method of starting an arc works satisfactorily with the more volatile metals it is not so successful with *pure* highly refractory ones.

Previous experiments * with iron electrodes have been successful, but in the present case, with very pure tungsten, the arc only started occasionally, and an applied potential difference of 200 volts was required, whereas with the more volatile metals acting as cathodes, 100 volts or less were sufficient. No external heat was applied to any electrode. When the arc did pass a brilliant pale blue coloured radiation was emitted which, on analysis, was found to contain the lines of hydrogen, mercury (vapour from the gauge), carbon dioxide, nitrogen, etc. (Pl. XX.), lines due to the residual gases in the tube. The spectrograms were photographed by means of a constant deviation spectrometer, the radiation from the arc stream being focussed on the slit.

The arc would not strike at a lower gas pressure than 10^{-2} mm. of mercury, and with an applied potential difference of 200 volts and a resistance in the circuit the potential difference across the arc electrodes was 40 volts with 5 amperes—this 40 volts being composed of the cathode fall, the fall in potential along the arc stream and the anode fall. It is much greater than in the cases of mercury, sodium, and other volatile metals. The cathode did not reach any state approaching incandescence, or even red heat—in fact, it remained comparatively cool, and so no tungsten lines appear in the spectrograms. On the other hand, the anode became red hot if the arc were allowed to continue for a few

* *Loc. cit.*

seconds, this being attributed to the bombardment of the positive electrode by electrons.

When a small bead of borax was fused on the cathode no difficulty was experienced in starting and maintaining the arc, even with 150 volts applied across the electrodes. The fall in potential in this case was 35 volts with 5 amperes and 30 volts with 5·5 amperes, this decrease of potential with increase of current being characteristic of arcs. The spectrum of the radiation (Pl. XX.) was very similar to that obtained with the pure tungsten electrodes except for the lines due to the presence of the borax. Again, in this case the cathode remained comparatively cool, whereas the anode quickly became red hot if the current were allowed to pass for a few seconds.

An examination of the spectrograms shows a little continuous spectrum but no tungsten lines, so it is evident that no part of the tungsten metal itself is raised to a temperature at which any appreciable vaporization occurs. The spectrograms were photographed as soon as the arc was started, and it was never allowed to continue for more than 3 seconds. The arc produced when borax was used on the cathode had a characteristic "flare" appearance and the borax quickly disappeared—the heat of the arc causing it to vaporize.

It is evident from these experiments that the starting and maintenance of these cold cathode arcs is due primarily to the presence of spots of impurities on the cathodic metal—at any rate in the case of highly refractory metals. This agrees with the evidence deduced by other experimenters from cold cathode currents drawn from small wires, that the emitting areas are very small. Thus the cathode current density at these small areas may be of the order required for Langmuir's hypothesis of strong field currents.

To explain the starting of the arcs with cold cathodes of highly refractory metals it seems that the initial momentary electrical discharge, giving a cathode fall in the neighbourhood of 100 volts, produces ions in the gas and the latter becoming more or less conducting, conditions are favourable for the arc current to pass. There is no question of thermionic emission, because the electrical discharge is only momentary and the arc strikes instantaneously, the high field within the cathode fall accelerating the electrons drawn from the very small emitting areas on the cathode.

LXXXVII. *Studies in the Spectra of High-Frequency Discharges. I.—Comparison of Damped with Undamped Electrodeless Excitation.* *By* J. K. Robertson, *Professor of Physics, Queen's University, Kingston, Canada* *.

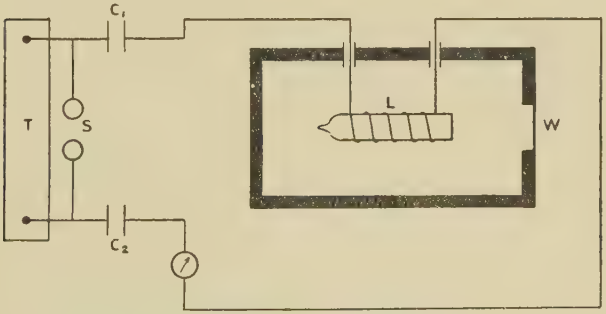
[Plates XXI.–XXIII.]

Introduction.

1. DISCHARGES in gases and vapours by the use of high-frequency currents may be obtained in a variety of ways. Outstanding among these are the following :—

(1) By the electrodeless method utilized in the pioneer work of Hittorf [1] and J. J. Thomson [2].

Fig. 1.



Damped wave circuit.

In this case the intensity of highly-damped oscillatory currents is controlled by the length of a spark-gap (see fig. 1). The advantages of this method in spectroscopic work have been indicated by the work of Kowalski [3], Hagenbach and Frey [4], Dunoyer [5], Robertson, McLennan, Bloch and Bloch, Déjardin, Balasse, Volkringer, Esclangon, Herzberg, Jung, Mierdel, Hart and Stuhlman † and others. In this paper this means of excitation will be described as D. W. (damped waves).

(2) By an electrodeless method in which the current traversing the exciting coil has its origin in the oscillations

* Communicated by the Author.

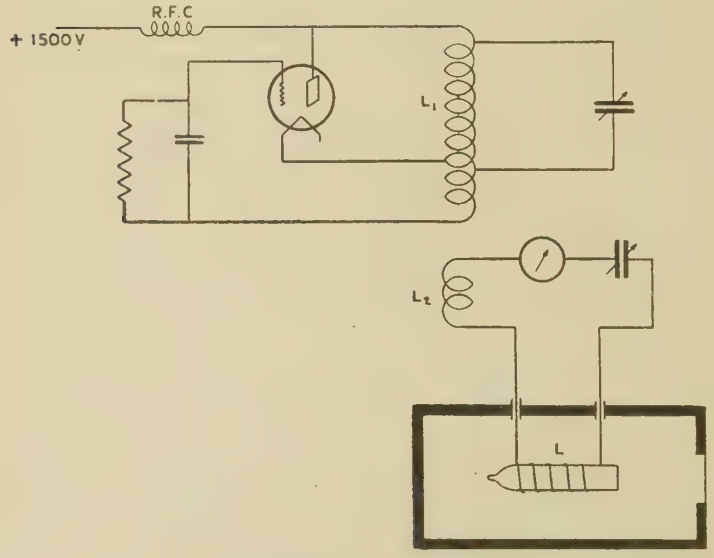
† See list of references at end of article, 6 to 27. As the method is illustrated by the work of almost any of these writers, no attempt has been made to make the list exhaustive.

of a valve circuit (see fig. 2). Reference will be made to this method as C.W. (continuous waves). The spectra of C.W. discharges have been observed by Clark [28], Ponte [29], Balasse [30], Knauss and Cotton [31], and others.

(3) By the use of external electrodes attached to two points on the coil carrying damped high-frequency currents. Examples of the use of this method will be found in the work of Chatterje [32] [33], Mukherji [34], and Ghosh and Chatterje [35].

(4) By the use of external electrodes, as in (3), but with undamped currents.

Fig. 2.



Continuous wave circuit.

C.W. excitation methods, both with and without electrodes, have been used in numerous recent investigations dealing, however, for the most part with the theory and mechanism of the discharge. The use of C.W. with electrodes for a study of the spectrum or the appearance of the discharge is found in the work of Heideman [36], Tolansky [37] [38], Okubo and Matuyama [39], Chenot [40], Tykocinski-Tykociner and Kunz [41], and others.

In view of the numerous uses to which high-frequency currents and circuits are nowadays being put, a careful

comparison of the above methods seemed desirable. By obtaining discharges in elements whose spectra are tolerably well-known, it was felt that differences in spectroscopic results might throw light on the mechanism of the discharge as well as prove of value in themselves. The work presented in this paper is restricted to a comparison of the first two methods, that is, the D.W. and the C.W. electrodeless methods. In a future communication it is hoped to present results dealing with the differences between C.W. discharges with electrodes and without.

Experimental.

2. The details of the standard D.W. circuit should be clear from fig. 1 in which T represents a transformer or large induction coil, S a spark-gap, C_1 and C_2 Leyden jars, and L a nickel solenoidal coil of some 15 turns within which was placed a tube of transparent fused quartz, about 2·5 cm. in diameter and 15 cm. long, containing either mercury or cadmium. The coil and the tube were inside an oven, heated electrically, with a quartz window W at the centre of one end. A radio-frequency ammeter in the oscillating circuit indicated currents of magnitude 2 to 2·5 amperes when a spark gap length of about 12 mm. was used. Wave-length measurements were not taken but the frequency was of the order of 10^6 cycles per second.

The C.W. circuit is shown in fig. 2. The upper half of the diagram represents a typical Hartley circuit involving an S.W. 2A Mullard transmitting valve, fed by a 1500 volt generator. The coil L_2 coupled with L_1 forms part of the oscillatory circuit used to excite the discharge. Coil L, the oven, and the quartz tubes were of course the same as in the D.W. arrangement. Wave-lengths of about 140 metres and currents (in coil L) of the order of 3·5 amperes were normal values.

Spectrograms were taken with a small Hilger quartz spectrograph, although a few others, not reproduced in this paper, were also taken in the visible region with a double prism Ladd instrument. Plates used were Wellington anti-screen, except when it was desired to examine carefully the region 1850–2200, when Schumann plates were substituted.

Results.

Mercury.

3. The spectrograms in Plate XXI. illustrate the important differences in the two types of discharge.

(*a*) By comparing *a* and *b* it will be seen that at low temperatures (vapour pressures less than 0·3 mm.) D.W. excitation brings out spark lines, whereas with C.W. these are either absent or present with feeble intensity. Note, for example, the absence of 1942, 2225, etc., Hg + lines in (*b*), as well as the reduced intensity of the low level Hg + line 2848. At low temperature D.W. excitation also brings out more readily the higher level sharp and diffuse series lines, although, strangely enough, the intercombination line 2856 ($2^3P_1 - 3'S$) is stronger in C.W.

[It is, of course, well-known, especially from the work of the French physicists cited above, that D.W. excitation, with varying spark-gap length and pressure may be used to classify the lines of an element, and objection may be made that, for a proper comparison of D.W. and C.W., D.W. spectra should also have been taken with a range of spark-gap lengths. To some extent this was done, as will be seen from an examination of Pl. XXII. where the spectra were taken with a spark length of 1 mm. and with exposure times of one hour. In this paper comparison with C.W. has been restricted to a gap of the order of one centimetre because (1) with this length at the low pressures, intensities of the same magnitude as for C.W. can be obtained with similar (and short) exposure times; (2) with this length the range of temperatures over which D.W. discharges can be obtained is comparable with that for C.W. (with a 1 mm. gap, a D.W. discharge could be obtained with difficulty above 160°); (3) the effect of changing the spark-length on D.W. discharge has already been examined in many investigations.]

(*b*) With rising temperature the spectra arising from the two types of excitation become much more alike. Compare, for example, *c* and *d*, taken at vapour pressures in the region 2·0 to 2·8 mm. In both, spark lines are absent; in both, higher series members are feebler or absent, coming out, if anything, better with C.W. than with D.W.

The three characteristic bands of *continua* *, (i.) to the long wave-length side of 2537, (ii.) a second with maximum intensity about 3340, (iii.) a third with maximum intensity in the visible, have now made their appearance, but with much greater intensity for C.W. excitation. The well-known mercury band, with head at 2345, is present in *d* but is absent or feeble in *c*. The presence of this band is an illustration of the necessity for exact information regarding different high-frequency discharges. Okubo and Hamada [39], for example, who worked with external electrodes, make the statement :—"The band at 2345 is lacking in the high-frequency excitations, in spite of the fact that it is strongly emitted in the branched and the distilled arcs."

At still higher pressures, from 11 to 15 mm., the same differences may be observed, but in a more marked degree, as is clearly shown in *e* and *f*. It is to be noted that at this stage the band 2345 is feebly present with D.W. excitation. In *f* there is also evidence of the presence of the 2480 band, although because of the small dispersion of the spectrograph and the nearness of the line 2482, the plates do not reveal exact details of the bands in this region.

(*c*) Observations were continued until temperatures as high as 240° (vapour pressure= 57 mm.) were reached. For the temperature range 210°–240° no D.W. spectrograms are reproduced because, although visible discharges were observed and several spectrograms taken, with exposures of the same order as for C.W., either no result was obtained or only the resonance line 2537 was recorded.

In *f* a typical exposure for C.W. at temperature 220° is given. The suppression of arc lines and the enhancement of the regions of *continua* are again evident. It is to be particularly noted that, with C.W. excitation, and sufficiently high vapour-pressure, continuous spectra can be photographed with exposures as low as 2′. Volkringer [42] in his memoir on continuous spectra (obtained with damped electrodeless excitation) calls attention to the fact that, in this same pressure region, wide slits and exposures of the order of an hour are necessary. In the work of Robertson, MacKinnon,

* For summary of work on continuous spectra up to 1930, see references 42, 43, at end of paper. Also reference 44.

and Zinn [9] exposures of several hours were necessary to bring out continuous regions, although, in Pl. XXII. of this paper, the exposures are only one hour.

Attention should be directed also to a *continuum* extending from the 2345 band to the region of shorter wave-lengths as far as 2190 *, which appear in the high density plates. From spectrogram *f* it will be noted also that the width of both this continuum and the 2345 band is only a fraction of that corresponding to a completely illuminated slit. This undoubtedly arises from the fact that at the end of the quartz tube remote from the spectrograph, the width of 2·5 cm. narrowed down to a 1 cm. piece by means of which connexion had originally been made to the exhaust system. In this narrow piece the discharge was more intense at high vapour-pressures, evidence no doubt of greater current density in this region.

Cadmium.

4. With changing temperature, general differences much the same as in mercury were observed with cadmium, as may be seen by an inspection of Pl. XXIII. The following features are of especial interest.

(*a*) The presence, at low temperature (vapour pressure 0·3 mm. or less), and with D.W. excitation, of the characteristic *pp'* lines, 2240, 2267, 2306, 2329.

(*b*) The ready self-reversal of the resonance line 2288 (1′S–2′P), with C.W. excitation, and the gradual development, with increasing pressure, of the continuous region associated with it.

With D.W. excitation and low pressure, a continuous region is observed, which, as may be noted in spectrum *a*, has a fair intensity in the interval 2265 to 2288 and extends on the long wave-length side to a limit difficult to determine. It is doubtful if this has the same origin as that of the continuous about 2288 which develops with increasing pressure.

With D.W. excitation, reversal of 2288 is evident only in spectrum *g*, that is, at the highest pressure, when the exposure was six hours. Particularly note-

* This no doubt is the same band as reported by Lord Rayleigh (Proc. Roy. Soc. A, cxix. p. 349, 1928) in the fluorescent spectrum of mercury when excited by the continuous spectrum of hydrogen.

worthy is the width of the reversed region in this plate for it covers about 10 angstroms.

(*c*) The sharpness of the resonance line 3261 ($1'S–2^3P_1$), even with C.W. excitation. Unlike the corresponding mercury line 2537, this cadmium line shows no sign of broadening or of the development of *continuum* near it until a temperature of 550° (pressure= 36 mm.) is reached. (See spectrum *h*.) Even in this case the broadening is slight.

(*d*) The emission of (i.) the continuous region around 2288 already mentioned; (ii.) the continuous region to be seen in spectrum *h*, extending from roughly 3800 to 4800; (iii.) in spectrum *a*, a narrow continuous band (marked with a X), of width about 16 angstroms, with centre at 2211 or 2212. Although this band does not show up clearly in the reproduction, on the original plate it can be seen with good intensity. This band at 2212 is undoubtedly the same as that observed in absorption by Mohler and Moore [45] and Winans [46]. Its presence in emission is another excellent illustration of the need for exact high-frequency data because Winans [46], using what is described as an electrodeless discharge (although a single external electrode attached to a low voltage Tesla coil was used) states that the 2212 band is *absent* in emission. Because of its absence in Winans' work and because the correctness of his theoretical interpretation depends to some extent on the presence of this band in absorption and its absence in emission, two years after spectrum *a* was taken observations were repeated. No difficulty was experienced in reproducing spectra in which the band appears in emission.

In passing it may not be amiss to note that Winans [46] reports the presence in emission of a band at 2114 which the author is unable to find on his plates.

(*e*) The absence of the cadmium forbidden lines 3320, 3141 ($1'S-2^3P_{0,2}$) in both D.W. and C.W. This fact is recorded because Okubo and Matuyama [39], working with external electrodes, report the presence of these lines in high frequency discharges.

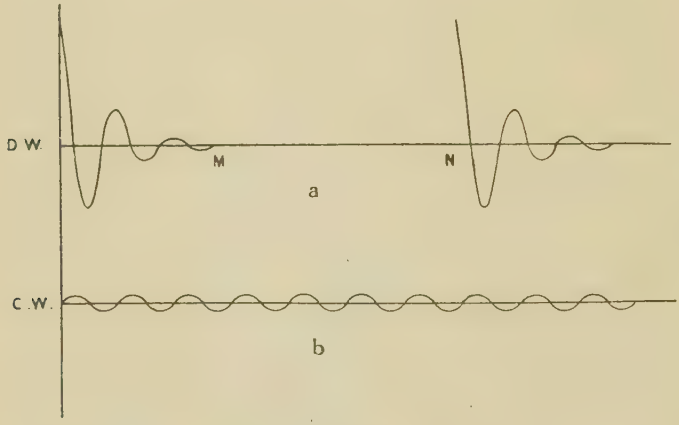
(A search was made for the corresponding mercury lines but owing to the low dispersion of the spectrograph and the nearness of other wave-lengths to the forbidden

lines of this element, it was difficult to be sure of their presence or absence.)

Discussion.

5. (*a*) The general differences between the two types of discharge may be ascribed to two causes :—(1) the much greater average electric intensity which exists during a single half-cycle with D.W. excitation than with C.W. ; (2) the comparatively long interval when D.W. currents do not exist. The differences between the two currents are roughly illustrated by the diagrams of fig. 3. Because of the highly damped D.W. currents, an ammeter in the oscillatory circuit tells nothing about

Fig. 3.



the average value of the current during a half-cycle. For an instrument reading of 2 or 3 amperes, the latter might easily be of the order of several hundred amperes. It is readily seen, therefore, why, at low pressures, when the mean free path of an atom is large, D.W. excitation, with spark-gap of length of several millimetres, is markedly different from C.W. During a single half-cycle an electron acquires high kinetic energy before impact takes place.

The decrease in the comparative effectiveness of D.W. excitation with increasing pressure undoubtedly lies in the long interval (MN, fig. 3) when no D.W. currents exist. Electrons, under C.W. excitation, are in constant to and fro motion, and the chances of an ionizing collision

at higher pressures, with decreased mean free path, are tremendously greater than with D.W. excitation. (At low pressures, because of the longer mean free path and the much smaller velocity of C.W. electrons compared with D.W. they may oscillate many times without making collisions).

For the same reason the bands of continuous spectra appear more readily with C.W. excitation. Although a final solution of all the regions of *continua* which may be observed in mercury and in other monatomic elements has not been given, the origin of at least some of these bands is in radiation from molecules consisting of the union of a normal atom and one excited to a resonance or a low-energy level. The *continuum* which gradually develops in the neighbourhood of the cadmium resonance line 2288, for example, may be ascribed to processes such as

$$Cd'_2 \rightarrow Cd + Cd + h\nu + \text{kinetic energy} + \text{energy of dissociation},$$

and $Cd'_2 = Cd_2 + h\nu$,

where Cd'_2 represents a molecular union of a normal atom and one in the 1′P state. Obviously high density and the existence of numerous excited atoms favour the formation of excited molecules. With C.W. excitation when electrons are continuously oscillating in a low electric field, the chances of formation of excited atoms must be enormously greater than with D.W.

(*b*) Attention has been directed to the broadened reversal, with D.W. excitation, of the 2288 cadmium resonance line, shown in spectrum *g*, Pl. XXIII. This is probably an example of the transition from the ordinary *atomic* absorption exhibited in self-reversal to absorption by molecules consisting of a union of loosely bound, unexcited atoms. No explanation can be offered as to why this broadening absorption occurs in spectrum *g* (D.W.) and not in *h* (C.W.). In seeking for a possible explanation along the lines suggested above, the writer ran into the difficulty that a similar broad absorption (of width about 7 angstroms) has been observed in certain conditions when C.W. excitation, with external electrodes, was used. It is hoped that the work now under way dealing with the comparison of C.W. discharges with and

without electrodes will throw further light on the mechanism of this broadened absorption.

(*c*) It has been noted above that, at low pressure, with C.W. excitation, the characteristic *pp'* group appears with strong intensity. To excite these lines an excitation potential about one-quarter of a volt above the ionization potential, that is, about 9·2 volts, is necessary. It is significant that, although the potential necessary to excite the *Cd* + resonance line 2265 is about 14·3 volts, the *pp'* fails to appear in any other spectrum, either C.W. or D.W., where 2265 is present. What then, are the peculiar conditions found in D.W. excitation at low pressures, which bring out this group ?

It is well known that the *pp'* lines originate when there is a simultaneous displacement of two electrons on excited orbits, and Ruark [47] has given evidence to show that a bombarding electron at a single impact may displace two electrons. Under the conditions in which the lines appear in this work, it is not considered likely that such a process occurs, for its probability would be much greater with C.W. excitation than with D.W. It seems reasonable to assume that the primary cause in this case is double ionization. On re-combination two electrons might readily be found on excited orbits.

Summary.

1. A comparison has been made of the spectra of high frequency discharges in mercury and cadmium vapours, obtained with damped currents (D.W. excitation) with those obtained by the use of oscillating valve circuits (C.W. excitation).

2. At low pressures, D.W. discharges (with spark-gap length of the order of 1 cm.) bring out spark and higher level arc lines better than C.W. At higher pressures C.W. discharges are more effective in bringing out higher series members.

3. At higher pressures, C.W. discharges are excellent sources of bands such as the mercury one with head at 2345, and also of regions of continuous spectra. The latter may be obtained with very short exposures.

4. In cadmium vapour, in disagreement with the work of Winans, a diffuse narrow continuous region with centre about 2212 has been observed in emission, with D.W. excitation, at low pressure.

5. A broadened self-reversal has been observed in cadmium, which is thought to be an example of a transition from atomic to molecular absorption.

6. A group of *pp'* lines in cadmium has been observed under conditions which suggest the possibility of recombination following double ionization as the mode whereby two electrons are on excited orbits.

Acknowledgment is gratefully made of the service rendered by Mr. Stewart Marshall, B.A., who, as research assistant, did most of the experimental work, and also that by Mr. C. W. Clapp, B.Sc., who took a few spectrograms and did the photographic enlarging. The writer would also like to express his gratitude to the Research Council of Canada for the loan of the motor-generator set which made possible the use of a valve circuit.

References.

(1) Hittorf, Wied. Ann. lii. p. 473 (1884).
(2) J. J. Thomson, Phil. Mag. xxxii. p. 321 (1891).
(3) Kowalski, *Phys. Zeit.* xv. p. 249 (1914).
(4) Hagenbach and Frey, *Phys. Zeit.* xviii. p. 544 (1917).
(5) Dunoyer, *Compt. Rendus*, clxxv. p. 350 (1922).
(6) Robertson, Phys. Rev. xix. p. 470 (1922).
(7) Robertson, Phil. Mag. i. p. 752 (1926).
(8) Robertson and Findlay, Trans. Roy. Soc. Can. xxi. p. 85 (1927).
(9) Robertson, MacKinnon and Zinn, J. O. S. A. xvii. p. 417 (1928).
(10) McLennan, Proc. Roy. Soc. lxxxvii. p. 256 (1912–13).
(11) Bloch and Bloch, *Journ. d. Phys.* iv. p. 333 (1923).
(12) Déjardin, *Ann. d. Phys.* ii. p. 322 (1924).
(13) Bloch, Bloch and Déjardin, *Ann. d. Phys.* ii. p. 461 (1924).
(14) Bloch, Bloch and Déjardin, *Journ. d. Phys.* vii. p. 129 (1926).
(15) Bloch and Bloch, *Ann. d. Phys.* xi. p. 141 (1929).
(16) Déjardin, *Journ. d. Phys.* vii. p. 129 (1926).
(17) Déjardin, *Ann. d. Phys.* viii. p. 424 (1927).
(18) Déjardin, *Ann. d. Phys.* xiii. p. 82 (1930).
(19) Balasse, *Compt Rendus*, clxxxiv. p. 1002 (1927).
(20) Balasse, *Compt. Rendus*, clxxxiv. p. 1320 (1927).
(21) Volkringer, *Compt. Rendus*, clxxxiii. p. 780 (1926).
(22) Volkringer, *Ann. d. Phys.* xiii. p. 15 (1930).
(23) Esclangon, *Journ. d. Phys.* vii. p. 52 (1926).
(24) Herzberg. *Ann. der Physik*, lxxxiv. p. 553 (1927).
(25) Jung, *Ann. der Physik*, lxxv. p. 201 (1924).
(26) Mierdel, *Ann. der Physik*, lxxxv. p. 612 (1928).
(27) Hart and Stuhlman, Phys. Rev. xxxv. p. 1437 (1930).
(28) Clarke, ' Nature,' cxx. p. 727 (1927).
(29) Ponte, ' Nature,' cxxi. p. 243 (1928).
(30) Balasse, *Compt. Rendus*, lcxxxiv. p. 1002 (1927).
(31) Knauss and Cotton, Phys. Rev. xxxviii. p. 1190 (1931).
(32) Chatterje, *Zeit. für Phys.* lxii. p. 9 (1930).
(33) Chatterje, *Zeit für Phys.* lxii. p. 712 (1930).
(34) Mukherji, *Zeit. für Phys.* lxiv. p. 698 (1930).
(35) Ghosh and Chatterje, *Zeit. für Phys.* lxv. p. 102 (1930).

(36) Heidemann, *Ann. der Physik*, lxxxv. p. 649 (1928).
(37) Tolansky, Proc. Phys. Soc. London, xlii. p. 556 (1929–30).
(38) Tolansky, Proc. Phys. Soc. London, xliii. p. 545 (1931).
(39) Okubo and Matuyama, Phys. Rev. xxxviii. p. 1651 (1931).
(40) Chenot, *Compt. Rendus*, cxcii. p. 38 (1931).
(41) Tykocinski-Tykociner and Kunz, Phys. Rev. xxxv. p. 1436 (1930).
(42) Volkringer, *Ann. d. Phys.* xiv. p. 15 (1930).
(43) Finkelnburg, *Phys. Zeit.* xxxi. p. 1 (1930).
(44) Nielson, Phys. Rev. xxxviii. p. 888 (1931).
(45) Mohler and Moore, J. O. S. A., xv. p. 74 (1927).
(46) Winans, Phil. Mag. vii. p. 555 (1929).
(47) Ruark, J. O. S. A. xi. p. 199 (1925).

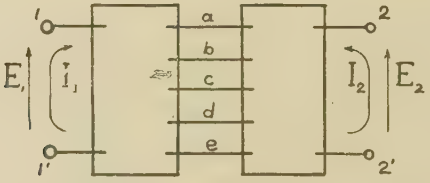
Department of Physics,
Queen's University,
Kingston, Canada,
July 1932.

LXXXVIII. *Note on Bartlett's Bisection Theorem for 4-Terminal Electrical Networks.* *By* O. Brune, *Sc.D.**

IN a paper by A. C. Bartlett (Phil. Mag. iv. p. 902 (Nov. 1927)) the following theorem is proved :—

If an artificial line section with characteristic impedance Z_0 and propagation constant θ has within itself terminals *a*, *b*, *c*, *d* . . . . such that a section through these terminals cuts the network into two exactly similar halves (fig. 1), and if each of these halves considered as a two-terminal impedance from the original line-section terminals has an impedance Z_1 when *a*, *b*, *c*, *d* are open, and an impedance Z_2 when *a*, *b*, *c*, *d* are short-circuited (fig. 2), then

Fig. 1.



$$Z_1 = Z_0 \coth \theta/2,\ Z_2 = Z_0 \tanh \theta/2.$$

This theorem has frequently proved useful, and the following shorter proof and generalization should be of interest.

* Communicated by the Author.

1. An artificial line-section with the bisection property is a symmetrical 4-terminal network (or, more properly, 2-terminal pair). The behaviour of such a network at its terminal pairs (1, 1′) (2, 2′) (see fig. 1) is characterized by a pair of equations:

$$\left.\begin{aligned} E_1 &= I_1 z_{11} + I_2 z_{12} \\ E_2 &= I_1 z_{21} + I_2 z_{22} \end{aligned}\right\}, \quad \ldots \quad (1)$$

which is equivalent to the pair

$$\left.\begin{aligned} E_1 &= E_2 \cosh\theta - I_2 Z_0 \sinh\theta \\ I_1 &= \frac{E_2}{Z_0}\sinh\theta - I_2 \cosh\theta \end{aligned}\right\}, \quad \ldots \quad (2)$$

the relations existing between the two sets of coefficients being readily found as

$$z_{22} = z_{11} = Z_0 \coth\theta, \quad \ldots \quad (3)$$

$$z_{21} = z_{12} = Z_0 \operatorname{cosech}\theta \quad \ldots \quad (4)$$

Fig. 2.

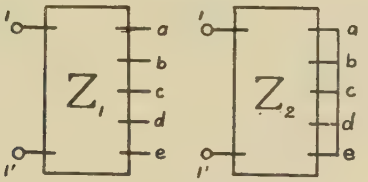
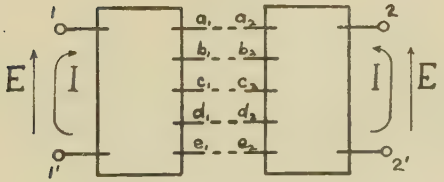


Fig. 3.



Now let equal voltages ($E_1 = E_2 = E$) be impressed across the terminal pairs causing equal currents ($I_1 = I_2 = I$) to flow. Then if the two halves be separated at $a, b, c, d \ldots$ equal voltages will appear across corresponding pairs of terminals ($a_1\, b_1$) a_2 b_2) .... (see fig. 3) because of the bisection property; hence no change takes place on reconnecting ($a_1\, a_2$) ($b_1\, b_2$) .... In other words, under the given conditions of impressed voltage no current flows at $a, b, c, d \ldots$ and the current flowing into the terminals (1 1′) say, is the same as would flow in the network Z_1 in fig 2.

Hence

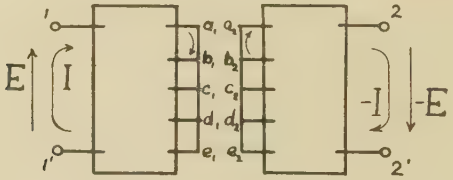
$$E = IZ_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and substituting in equation (1) $E_1 = E = IZ_1$, $I_1 = I_2 = I$ we get

$$Z_1 = z_{11} + z_{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Again, let voltages of equal magnitude, but opposite direction, be impressed across the terminal pairs (1 1′) (2 2′). Now, if the terminals a, b, c, d be short-circuited and the two halves separated, equal and opposite currents would appear in corresponding short-circuiting leads. If, therefore, the two halves are reconnected, the resultant current in the short-circuiting leads would be zero, and no change would take place if they were removed ; in other words, no voltage exists across the terminals a, b, c, d under these conditions

Fig 4.



of impressed voltage. Hence if I′ be the current entering at (1 1′)

$$E = I' Z_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and substituting in equation (1) $E_1 = I'Z_2$, $I_1 = -I_2 = I'$ we get

$$Z_2 = z_{11} - z_{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The results (6) and (8) contain the theorem for, substituting from equations (3) and (4) in (6) and (8),

$$Z_1 = Z_0 (\coth \theta + \operatorname{cosech} \theta) = Z_0 \coth \theta/2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$Z_0 = Z_0 (\coth \theta - \operatorname{cosech} \theta) = Z_0 \tanh \theta/2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

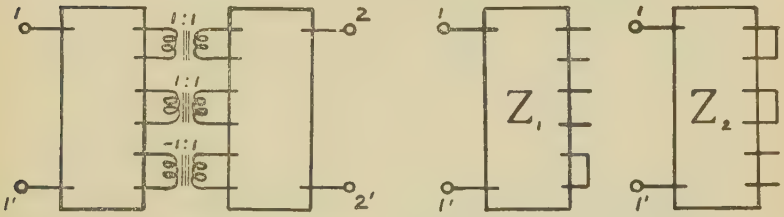
2. In the above proof it becomes evident that complete similarity in the internal structure of the two halves is not essential; it is necessary only that their external behaviour with respect to the terminal-pairs of the original network and the terminals at the section (a, b, c . . .) be the same.

If the artifice of connecting the two halves through ideal transformers of 1 : 1 ratio be resorted to, then again, for the

bisection theorem to hold, equivalent performance in these halves will be required only with respect to the terminal-pairs of the original network and the *terminal-pairs* connected through the ideal transformers. Note, however, that in Z_2 of the theorem only terminal-pairs joined through ideal transformers will be short-circuited.

Finally, if some of these ideal transformers have the ratio $-1:1$ instead of $1:1$ the necessary modification in the bisection theorem is readily seen to be the following :— Z_1 is now the impedance of the two-terminal network which is formed when in one of the halves the terminal-pairs across which ideal transformers of $-1:1$ ratio appear are short-circuited, the others being left open ; Z_2 is the corresponding impedance when the terminals across which ideal transformers of $1:1$ ratio appear, are short-circuited, the rest being left open (see fig. 5)*.

Fig. 5.

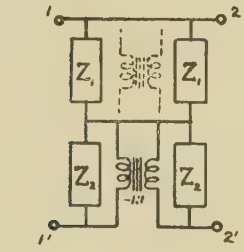


This generalization is interesting in that it links the bisection theorem with the networks of fig. 6, which by suitable choice of Z_1 and Z_2 can be made equivalent to any symmetrical two-terminal pair †. The passage from the structures of fig. 6 (*a*), (*b*), (*c*), (*d*) to the simpler structures of fig. 6 (*e*), (*f*), (*g*), (*h*) respectively is fairly obvious. It may be remarkod that there is no essential difference between figs. 6 (*b*), (*c*), and (*d*), except in their arrangement for subsequent simplification to fig. 6 (*f*), (*g*), (*h*) respectively. In fig. 6 (*f*) the two elements Z_1 can be combined in parallel, since the ideal transformer forces the currents in them always to be the same. In fig. 6 (*d*) the transformer of $-1:1$ ratio may be replaced by a mere crossing of leads because of the special arrangement of the structure.

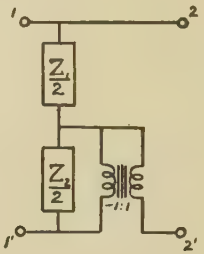
* It is obvious that at least one of the transformers of $+1:1$ ratio may be omitted.

† *Cf.* Cauer, *Sitzungsber. d. Preuss. Akad. d. Wissensch.*, Dec. 1927.

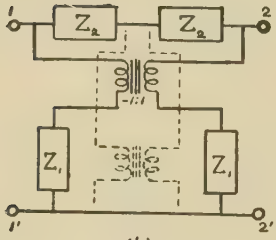
Fig. 6.



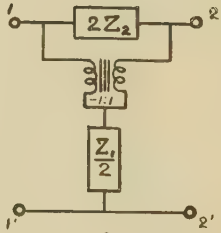
(a)



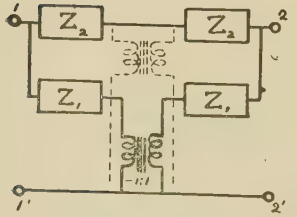
(e)



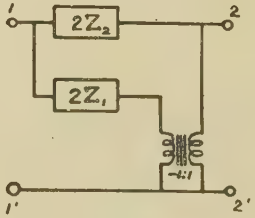
(b)



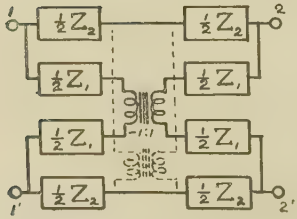
(f)



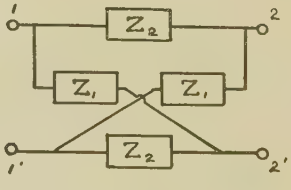
(c)



(g)



(d)



(h)

The lattice structure of fig. 6 (*h*) is well known, while the structures of fig. 6 (*e*), (*f*) have also recently appeared in the literature *.

LXXXIX. *Some Improvements in the Use of the Lees and Chorlton Heat Conductivity Apparatus.* *By* CHARLES H. LEES, *D.Sc.*, *F.R.S.*, *and* H. A. NANCARROW, *B.Sc.*, *Research Student, East London College, University of London* †.

1. THE object of the method developed by Lees and Chorlton ‡ for the determination of the heat conductivities of materials in use in industry was to simplify as much as possible the apparatus and the observations required in making a determination without too great a sacrifice of accuracy in the final result. When the constants of the apparatus had been determined the only observations required were the thickness of a disk of the material to be tested, the temperatures of two metal disks between which the material was placed, and the temperature of the air in the neighbourhood of the apparatus.

Experience in the use of the apparatus has led us to make some changes in the method of using it, and in the method of determining its constants and calculating the conductivity from the observations, which we consider add materially to its convenience and accuracy.

The apparatus still consists of two circular brass plates 12 cm. diameter and 1·2 cm. thick, between which a circular disk of the material to be tested, of the same diameter and a few millimetres thick, is placed. The upper brass plate forms the bottom of a metal box about 3 cm. deep through which steam can be circulated. The plates are suspended in air, with their flat surfaces horizontal, by three strings attached to three screws in the edge of the lower plate.

The support for the strings should be such as to allow the air in the neighbourhood free access to the lower plate, and, as a rule, the temperature of this air is least influenced by the apparatus if the lower plate is about 4 feet above the floor of the room and no furniture interferes with the movement of the air from the floor to the plate, or raises the temperature

* H. G. Baerwald, *Sitzungsber d. Preuss. Akad. d. Wissensch.* (1931).
† Communicated by the Authors.
‡ Philosophical Magazine, xli. p. 495 (1896).

of that air by getting heated itself by radiation from the apparatus. A thermometer with its bulb protected by a small bright metal screen from radiation from the apparatus is placed horizontally about 30 cm. below the plate to give the temperature of this ascending air.

The temperature of each brass disk is determined by a thermometer, graduated in half degrees or in tenths, placed in a radial hole in the mid plane of such depth that the centre of the bulb is about 3 cm. from the centre of the disk, the length of the bulb itself not exceeding 2 cm. The bulbs are smeared with glycerine so as to make good thermal contact with the metal, and short lengths of thin rubber tubing on the stems hold the thermometers securely in the holes.

The readings of the three thermometers are the only observations necessary for the determination of the heat conductivity of the material of a disk of measured thickness, and the constants of the apparatus are determined by a cooling experiment in which the three thermometers and the time are observed.

2. In the original paper it was stated that the under surface of the lower disk could be either polished or be painted, but we have found it better to varnish the lower surface and edge of the lower disk, and the edge of the specimen to be tested, with a celluloid varnish which neither becomes tacky nor discoloured by heating to 100° C. The edge of the upper disk and the edge and top surface of the heater are covered with a closely-fitting cylinder of insulating material, *e. g.* woollen cloth, which can slide up or down the vessel in order to expose more or less of the surface to cooling air currents.

The contact between the brass disks and the material under test was in the original made by a mercury film, but the difficulty of keeping the brass surfaces sufficiently clean to give a mercury mirror and enable the contact with the material to be made without including any air film, has led us to substitute glycerine for the mercury and to make a small correction for the thermal resistance of the glycerine film, determined by an experiment which gives at the same time the thermal resistances of the brass plates between their surfaces in contact with the glycerine and horizontal planes through the centres of the thermometers. These brass plate resistances were taken as negligible in the theory given in the original paper.

The principle of the method is the use of the loss of heat

from the surface of the lower disk to determine the flow of heat through the material, and from this and the difference of temperature of its surfaces to calculate the conductivity.

3. In previous work on flow of heat it has been customary to take the loss of heat from a heated solid to surrounding gases to be proportional to the excess of the temperature of the surface-losing heat over that of the gas, that is to use Newton's law of cooling. It is, however, well known, that the loss of heat increases faster than the excess of temperature at a rate equal to about the 5/4th power of that excess, but the difficulty of dealing with the equations which that law introduces has prevented its use except in the case of an infinitely long bar heated at one end and in calorimetrical cooling corrections *. We propose to show how this law may be utilized to render the calculations more accurate.

4. *Theory of the temperature distribution in a heated horizontal circular disk.*—If the temperature excess over that of the air at each point of a horizontal plane at height x above some convenient level be v, and if q is the area and p the length of the perimeter of the horizontal section of the disk, k the heat conductivity of the material of the disk, and hv^n the loss of heat per second per square centimetre of surface exposed to the air, the equation satisfied by v throughout the disk is

$$\frac{d^2v}{dx^2} = \frac{ph}{qk} v^n. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.1)$$

As the apparatus is designed so as to limit the variations of v to the range 70° to 80° C. excess above air temperature, we may write $v = v_1\left(1 + \frac{v}{v_1} - 1\right)$, where v_1 is some convenient temperature within this range, and then

$$hv^n = hv_1{}^n\left\{1 + n\left(\frac{v}{v_1} - 1\right) + \frac{n(n-1)}{2}\left(\frac{v}{v_1} - 1\right)^2 + \ldots\right\}$$

$$= hnv_1{}^{n-1}\left\{v - \frac{n-1}{n} v_1 + \frac{n-1}{2}\frac{(v-v_1)^2}{v_1} + \ldots\right\},$$

and neglect the third term in the bracket whose value is of the order 1/400.

Writing h_1 for $hnv_1{}^{n-1}$ equation (4.1) becomes

$$\frac{d^2v}{dx^2} = \frac{ph_1}{qk}\left(v - \frac{n-1}{n} v_1\right), \quad . \quad . \quad . \quad (4.2)$$

* Both used for some years at this College. See Schuster and Lees, Exercises in Practical Physics,' 5th edition, p. 116 (1925).

and its solution is

$$v-\frac{n-1}{n}v_1 = \mathrm{A}\sinh\sqrt{\frac{ph_1}{qk}}\,.\,x+\mathrm{B}\cosh\sqrt{\frac{ph_1}{qk}}\,.\,x,$$

where A and B are constants of integration.

If x is measured from a plane at which the temperature excess is v_0 and the total heat flow through it per second down the temperature slope is H_0,

$$\mathrm{A}=\frac{\mathrm{H}_0}{\sqrt{ph_1qk}} \quad \text{and} \quad \mathrm{B}=v_0-\frac{n-1}{n}v_1,$$

and the solution becomes

$$u=u_0\cosh\alpha x+\frac{\mathrm{H}_0}{\sqrt{ph_1qk}}\sinh\alpha x,$$

and the total heat flow per second down the temperature slope is

$$\mathrm{H}_x=qk\frac{dv}{dx}=\mathrm{H}_0\cosh\alpha x+v_0\sqrt{ph_1qk}\,.\sinh\alpha x,$$

where, for convenience, u is written for $v-\frac{n-1}{n}v_1$ and α^2 for $\frac{ph_1}{qk}$.

If the medium considered of conductivity k_a extends from $x=-a$ to $x=a$, and at both these planes is in contact with disks of a material of conductivity k_b, the conditions to be fulfilled at the interfaces are that temperatures and heat flows shall be continuous.

Hence, since at $x=a$ we have

$$u_a=u_0\cosh\alpha a+\mathrm{H}_0\mathrm{R}_a\mathrm{Sh}\alpha a$$

and

$$\mathrm{H}_a=\mathrm{H}_0\cosh\alpha a+u_0ph_1a\,\mathrm{Sh}\alpha a,$$

where $\mathrm{Sh}\alpha a$ is written for $\frac{\sinh\alpha a}{\alpha a}$ and R_a for $\frac{a}{qk_a}$ the thermal resistance of the disk of the k_a material, we have in the disk k_b with β^2 written for ph_1/qk_b

$$u_x=u_a\cosh\beta(x-a)+\mathrm{H}_a\frac{x-a}{qk_b}\,\mathrm{Sh}\beta(x-a).$$

and

$$\mathrm{H}_x=\mathrm{H}_a\cosh\beta(x-a)+u_aph_1(x-a)\mathrm{Sh}\beta(x-a).$$

If the k_b disk extends to the plane $x=a+b$ we have at that plane

$$u_b = u_a \cosh \beta b + \mathrm{H}_a \mathrm{R}_b \mathrm{Sh}\, \beta b$$

and

$$\mathrm{H}_b = \mathrm{H}_a \cosh \beta b + u_a p h_1 b \,\mathrm{Sh}\, \beta b,$$

where R_b is written for $\frac{b}{qk_b}$ the thermal resistance of the disk of k_b material.

In the same way, if beyond the plane $x=a+b$ there is a disk of conductivity k_c extending to the plane $x=a+b+c$, at that plane we have

$$u_c = u_b \cosh \gamma c + \mathrm{H}_b \mathrm{R}_c \mathrm{Sh}\, \gamma c$$

and

$$\mathrm{H}_c = \mathrm{H}_b \cosh \gamma c + u_b p h_1 c \,\mathrm{Sh}\, \gamma c,$$

where R_c is written for $\frac{c}{qk_c}$ the thermal resistance of the disk of k_c material, and $\gamma^2 = ph_1/qk_c$.

If the pile of disks is symmetrical about the first disk we get the temperatures u_{-a}, u_{-b}, and u_{-} and the heat flows H_{-a}, H_{-b}, and H_{-c} at the interfaces on the negative side of the origin by changing the signs of a, b, and c in the corresponding quantity for the positive without changing the signs of the thermal resistances R_a, R_b, R_c.

5. From these equations we may eliminate all but three of the interfacial temperatures and heat flows; for instance, all but two temperatures and a heat flow, and we may select these to suit the circumstances. In the apparatus under consideration it is most convenient to eliminate all but the temperatures u_c and u_{-c} and the heat flow H_{-b}.

The result is

$$\mathrm{R}_a \mathrm{Th}\alpha a + \mathrm{R}_b \mathrm{Th}\beta b + \mathrm{R}_c \mathrm{Th}\gamma c + p h_1 b \,\mathrm{Th}\beta b \,.\, \mathrm{R}_a \mathrm{Th}\alpha a \,.\, \mathrm{R}_c \mathrm{Th}\gamma c$$

$$= \frac{u_c - u_{-c}}{2} \cdot \frac{1 + p h_1 b \,.\, \mathrm{Th}\beta b \,.\, \mathrm{R}_a \mathrm{Th}\alpha a}{\cosh \gamma c \,.\, \mathrm{H}_{-b} + p h_1 \dfrac{(a \mathrm{Th}\alpha a + b \mathrm{Th}\beta b)(u_c + u_{-c})/2}{1 + p h_1 \{a \mathrm{Th}\alpha a (\mathrm{R}_b \mathrm{Th}\beta b + \mathrm{R}_c \mathrm{Th}\gamma c) + b \mathrm{Th}\beta b \,.\, \mathrm{R}_c \mathrm{Th}\gamma c\}}},$$

where $\mathrm{Th}x$ is written for $\tanh x/x$.

6. In the apparatus as used the values of $\mathrm{Th}\beta b$, $\mathrm{Th}\gamma c$, and $\cosh \alpha a$ differ from unity by quantities of the order 0·0001,

while Thαa only differs from unity by 0·001 when the thermal conductivity of the material tested is very low and the disk of it is unnecessarily thick. In almost all cases likely to arise each Th function may be taken as unity, and as products of them only occur in small correcting terms these products may also be taken as unity.

The equation thus simplified becomes

$$\mathrm{R}_a+\mathrm{R}_b+\mathrm{R}_c+ph_1b\,\mathrm{R}_a\mathrm{R}_c=\frac{u_c-u_{-c}}{2}\cdot\frac{1+ph_1b\,\mathrm{R}_a}{\mathrm{H}_{-b}+\dfrac{ph_1(a+b)(u_c+u_{-c})/2}{1+ph_1\{(a+b)\mathrm{R}_c+a\mathrm{R}_b\}}}.$$

The term $ph_1b\,\mathrm{R}_a\mathrm{R}_c$ is less than 0·0001 of the first term on the left, and terms of the type $ph_1(a+b)\mathrm{R}_c$ are less than 0·001. Hence, to the order of accuracy required, we may write the equation

$$2\mathrm{R}_a+2\mathrm{R}_b+2\mathrm{R}_c=\frac{v_c-v_{-c}}{\mathrm{H}_{-b}+p(a+b)h\left(\dfrac{v_c+v_{-c}}{2}\right)^n},$$

or, since in the case the lower disk is exposed to the air $\mathrm{H}_{-b}=(q+2cp)hv_{-c}{}^n$,

$$2\mathrm{R}_a+2\mathrm{R}_b+2\mathrm{R}_c=\frac{(v_c-v_{-c})}{\left\{(q+2cp)hv_{-c}{}^n+(a+b)ph\left(\dfrac{v_c+v_{-c}}{2}\right)^n\right\}}.$$

The thickness b of each glycerine film will, if the disks are put together by sliding, be very small—of the order 0·001 cm., while a is of the order 0·5 cm. Hence b may be neglected in comparison with a, but R_b is generally greater than $0{\cdot}01\mathrm{R}_a$ and cannot be neglected.

By substituting for the material under test a thin sheet of metal foil, whose thickness and thermal resistance may be neglected, we have, if v_c', v_{-c}' are then the temperatures of the brass disks,

$$2\mathrm{R}_b+2\mathrm{R}_c=\frac{(v_c'-v_{-c}')}{(q+2cp)hv'^n}.$$

On subtraction we have

$$\frac{2a}{qk_a}=\frac{v_c-v_{-c}}{(q+2cp)hv_{-c}{}^n+aph\left(\dfrac{v_c-v_{-c}}{2}\right)^n}-\frac{v_c'-v'_{-c}}{(q+2cp)hv'_{-c}},$$

which determines k_a when the constants q, p, c, and h of the apparatus are known and v_c, v_{-c} for the material of thickness $2a$ are observed, and $v_c' + v_{-c}'$ for the metal foil are known.

7. *Determination of the coefficient h of loss of heat from the surface of the apparatus to the air.*—As the orientation of a surface with respect to the air currents influences the cooling of the surface it is necessary in determining h by allowing the lower disk to cool in air, that the apparatus should be arranged as it was in the conductivity observations, and that the upper surface of the disk should be protected in some way to ensure that the loss or gain of heat by the disk through this surface is small and calculable from the dimensions and properties of the materials in contact with the surface. This involves superposing on the surface a disk of a low conducting material surmounted by the upper metal disk at a known temperature, so that the flow through the material can be calculated. The most convenient material is air, and if it is used it must be confined within a ring of a solid low conductor, and to prevent convection currents in the enclosed air the upper metal disk must be at a higher temperature than the lower, and the temperature of the ring must rise continuously from the lower to the upper.

The heat lost from its side surface to the air around the apparatus produces heat stream lines near the outer surface of the ring which are not perpendicular to the flat surfaces of the metal plates, but if the ring has the excess of its outer over its inner diameter three or four times its thickness, the stream lines near the inner surface are perpendicular to the plane surfaces.

Since the heat communicated to the lower disk through the air and ring is small compared to the loss from the surface of the disk to the air around it, simple methods of calculating it will give its value with sufficient accuracy *.

In the first instance the difference between the temperatures of the two surfaces of the metal disks and those of the thermometers in the disks will be neglected, as they are only about 0·01° C. In the second the effect of the glycerine layers will be allowed for by adding 0·002 cm. to the thickness of the solid ring, as the difference between the heat conductivity of glycerine and the material of the ring will not be large.

* This flow of heat into the lower disk was taken as negligible in the original paper of 1896.

The heat transmitted per second through the middle plane between the two metal surfaces will therefore be

$$\left[q_a\left\{\frac{k_a}{2a}+\sigma\frac{(v_c+v_{-c})^3}{2}\right\}+(q-q_a)\frac{k_r}{2a}\right](v_c-v_{-c}),$$

where k_a, k_r are the conductivities of air and ring, q_a the section of the air layer, σ the radiation constant, v_c and v_{-c} the absolute temperatures of the metal disks, and $2a$ the thickness of the ring and glycerine layers.

Of this heat a quantity $aph(v_{-c}-\mathrm{V})^n$ is lost from the outer surface of the ring below the mid plane, where V is the temperature of the air around. Hence the net flow of heat per second into the lower disk is

$$\left[q_a\left\{\frac{k_a}{2a}+\sigma\frac{(v_c-v_{-c})^3}{2}\right\}+(q-q_a)\frac{k_r}{2a}\right](v_c-v_{-c})-aph(v_{-c}-\mathrm{V})^n.$$

8. If m is the mass, C the specific heat of the material of the lower disk and the small difference between the mean temperature of the material of the disk, the mean temperature of its surface and the temperature of the thermometer in the disk are neglected. The equation for the cooling of the lower disk is

$$m\mathrm{C}\frac{dv_{-c}}{dt}=-(q+2cp)h(v_{-c}-\mathrm{V})^n+\mathrm{A}(v_c-v_{-c})-aph(v_{-c}-\mathrm{V})^n,$$

where A is written for the mean value of

$$q_a\left\{\frac{k_a}{2a}+\sigma\frac{(v_c+v_{-c})^3}{2}\right\}+(q-q_a)\frac{k_r}{2a}.$$

It may sometimes be desirable by proper choice of a and v_c-v_{-c} to make the mean value of

$$\mathrm{A}(v_c-v_{-c})-aph(v_{-c}-\mathrm{V})^n$$

zero or very small, so that this term may be neglected, but as there will be cases in which this cannot be done we retain both terms and write

$$m\mathrm{C}\frac{dv_{-c}}{dt}=-(q+2cp+ap)h(v_{-c}-\mathrm{V})^n+\mathrm{A}(v_c-v_{-c}).$$

It is not generally possible, without doing away with the simplicity of the apparatus, to secure that V the temperature

of the air and $v_c - v_{-c}$ the difference of temperatures of the metal disks are constants during the 20 minutes of a cooling experiment : and, in consequence, the direct integration of the equation, so as to give v_{-c} as a function of t, is not possible. Since, however, observations of the three temperatures may be made throughout the cooling, h may be determined from them by approximation.

Integrating the equation between the limits t' and t we have

$$(q+2cp+ap)h\int_{t'}^{t}(v_{-c}-\mathrm{V})^n\,dt = m\mathrm{C}(v_{-c}'-v_{-c})+\mathrm{A}\int_{t'}^{t}(v_c-v_{-c})\,dt.$$

Hence

$$h=\frac{m\mathrm{C}(v_{-c}'-v_{-c})+\mathrm{A}\int_{t'}^{t}(v_c-v_{-c})\,dt}{(q+2cp+ap)\int_{t'}^{t}(v_{-c}-\mathrm{V})^n\,dt}$$

$$=\frac{m\mathrm{C}}{q+(2c+a)p}\cdot\frac{v_{-c}'-v_{-c}+\dfrac{\mathrm{A}}{m\mathrm{C}}\displaystyle\int_{t'}^{t}(v_c-v_{-c})\,dt}{\displaystyle\int_{t'}^{t}(v_{-c}-\mathrm{V})^n\,dt}.$$

In case $v_c - v_{-c}$ is constant throughout the interval the integral in the numerator is replaced by $\mathrm{A}(v_c-v_{-c})(t-t')$, and in case $\mathrm{A}(v_c-v_{-c})-aph(v_{-c}-\mathrm{V})^n$ is very small, A is put $=0$ in the numerator and a put $=0$ in the denominator of the right-hand side of the equation.

If, in the latter case, V is also constant the cooling equation reduces to

$$m\mathrm{C}\frac{dv_{-c}}{dt}=-(q+2cp)h(v_{-c}-\mathrm{V})^n,$$

which gives on integration between the same limits

$$(v_{-c}-\mathrm{V})^{1-n}-(v_{-c}'-\mathrm{V})^{1-n}=(n-1)\frac{q+2cph}{m\mathrm{C}}(t-t'),$$

where v_{-c}, v_{-c}' are the temperatures of the disk at t and t' respectively. For brass of composition 60 Cu 40 Zn, $\mathrm{C}=0{\cdot}0917$ at the average temperature of the lower disk. For brass of other composition a determination of C is advisable.

In all cases, therefore, the value of h required in the

equation for the conductivity equation is determined, and may be considered as a constant of the apparatus as set up.

9. Observations on the heat given up by a heated body to a gas in which it is immersed and in which it causes convection currents*, have led to the conclusion that the heat transfer per square cm. per second is proportional approximately to the 5/4th power of the temperature difference between the body of the gas and the surface of the solid. For bright metal rods of diameter about 2 cm. the power was found to be 1·21 for temperatures up to 100° C †. If the heat given to the gas by a hot surface is approximately the same as that taken from the gas by a cool surface for the same temperature differences, and n is taken as p/q so that $-(dv/dt)=v^{p/q}$, where v is the temperature difference, then, on writing the equation $(-dv/dt)^q=v^p$, we see that p and q must both be odd integers and $n=9/7$ or $=11/9$ are possible values, *i. e.* $n=1{\cdot}29$ or $=1{\cdot}22$.

Since, in what follows, the temperature excess varies over a small range 85° to 50°, n is taken for convenience of calculation 5/4 and Barlow's 'Tables of Squares, etc.' are used for the 1/4th root.

The ring of low conductivity between the disks may be cork, cardboard, or wood, with its outer edge varnished with the same varnish as the disks. Its flat surfaces should be smeared with a little glycerine, and when between the disks the latter should be pressed together with a slight screwing motion to reduce the thickness of the glycerine layer as much as possible.

10. The apparatus having been assembled, steam is passed through the chamber of the upper disk while the lower has placed beneath it a sheet of mica supported on wire gauze, underneath which is a bunsen flame. When the lower disk has been raised to 102° or 103° C. the flame and mica are withdrawn and the disk allowed to cool till it is about 0·5° C. above that of the upper disk, when the supply of steam to the latter is cut off and any condensed steam in the heater run out by tilting the heater. On placing the disks horizontal again it is generally found that the upper disk is 0·2° to 0·4° C. hotter than the lower, and as the two cool this difference is maintained. If it is exceeded the heat insulation of the upper disk and heater must be slightly

* See A. H. Davies, Phil. Mag. xliv. p. 920 (1922) and the references given there.

† Lees, Phil. Mag. xxviii. p. 429 (1889), $0{\cdot}000098\ (v-V)^{1{\cdot}21}$.

reduced by sliding the insulating cap upwards, while if the upper disk cools faster than the lower the heat insulation must be increased or a little of the condensed steam must be allowed to remain in the heater when it is tilted. As the former alternative is the more convenient if the apparatus is to be used for many tests in succession, it is better to adjust the thickness of the insulating cap of the upper disk so that a small movement of the cap up or down keeps the temperature of the upper a few tenths higher than the lower.

It is convenient to take the temperature of the upper disk at each minute, that of the lower disk at each of minute, and that of the air as quickly as possible afterwards. About 20 minutes cooling should be observed, and the observations may, for convenience of calculation, be arranged in two groups, as in the table which follows.

11. A cooling observation (see Table 11).

In the apparatus, as set up, the ring was of thin cardboard 0·120 cm. thick, and the difference between its outer and inner radii was 0·50 cm.

Hence $q=113$ sq. cm., $q_a=95$ sq. cm., and $q-q_a=18$ sq. cm., k_r for the cardboard is taken as 5×10^{-4}, and to allow for the two glycerine layers 0·002 cm. is added to the thickness which becomes 0·122 cm.

The heat conductivity of the air enclosed is taken as $5{\cdot}7\times10^{-5}$ and the radiation constant as $1{\cdot}28\times10^{-21}$ gram degrees per sq. cm. per second per unit of difference of the 4th powers of the absolute temperatures of the two surfaces, as the radiation in the enclosed space between the disks may be taken as complete radiation.

The mean value of A is thus

$$95\left(\frac{5{\cdot}7}{{\cdot}122}10^{-5}+1{\cdot}28\times10^{-12}\,.\,706^3/2\right)+18\frac{5}{{\cdot}122}10^{-4}=0{\cdot}14,$$

and the mean value of $A/mC=0{\cdot}0014$. In the cooling observation to which the table refers, the mean value of v_c-v_{-c} was 0·52° C., so that $A(v_c-v_{-c})$ had the mean value of 0·073 gram degrees per second, while the mean value of

$$aph(v-V)^n=0{\cdot}061\times38\times9\times10^{-5}(60)^{5/4}$$

$=0{\cdot}035$ gram degree per second, each being calculated for the mean temperatures during the cooling.

The mean difference $A(v_c-v_{-c})-aph(v_{-c}-V)^{5/4}$ during the cooling was thus$=0{\cdot}038$ gram degrees per second, and for many purposes the two terms might be neglected in

comparison with the mean loss of heat from the lower disk, which is 2·5 gram degrees per second.

If it is intended to neglect them they should be made more nearly equal by using a thicker ring or a smaller difference of temperature between the two disks.

12. Table of Conductivi

$$\text{ermal resistance} = \frac{2a}{qk_a} = \frac{v_c - v_{-c}}{\left\{(q+2cp)v_{-c}^{5/4} + ap\left(\frac{v_c+v_{-c}}{2}\right)^{5/4}\right\}h} - \frac{v_c' - v_{-c}'}{(q+2cp)hv'_{-c}{}^{5}}$$

to the observations with tin foil. $q = 113$ sq. cm., $2c$

Material.	Thickness, cm.	Temperature. Upper.	Lower.	Air.	$v_c - v_{-c}$.	v_{-c}.	$v_{-c}{}^{5/4}$.	$158\ v_{-c}{}^{5/4}$.	$\frac{v_c+v_{-c}}{2}$.
		°C.	°C.	°C.		°C.			
in foil.....	·003	96·45	96·25	18·50	0·20	77·75	230·9	36480	77·85
lass A.....	·336	97·00	92·40	18·80	4·60	73·60	215·6	34070	75·90
lass B.....	·318	95·55	90·85	16·00	4·70	74·85	220·2	34790	77·20
ate	·334	96·10	93·95	18·40	2·15	75·55	222·7	35190	76·62

12. Conductivity observations, see Table 12.

The two disks of glass were cut from two sheets of window glass from different sources, and the results may be taken as representative for the glass now used for windows. The conductivity found for rather thinner window glass in the 1896 paper was 0·00203, but both the terms considered negligible in that paper, but included in the present treatment, see sections 2 and 7, would, if taken into account, have raised the value. The disk of slate was cut from a school slate, and the temperatures observed showed directly that it was a better conductor than the school slate used in the 1896 apparatus.

From the observed temperature falls through the three disks it is seen that the results would have been improved in accuracy if somewhat thicker disks had been used, particularly in the case of slate.

$$\text{Column E} = ap\left(\frac{v_c+v_{-c}}{2}\right)^{5/4},$$

$$\text{,,} \quad \text{F} = (q+2cp)v_{-c}{}^{5/4} + ap\left(\frac{v_c+v_{-c}}{2}\right)^{5/4}.$$

13. It will be seen from the above table that once the constants of the apparatus have been determined by the cooling experiment and by the experiment with the tin foil, the determination of the heat conductivity of a material is made by a measurement of the thickness of a disk and

ERVATIONS AND RESULTS.

first fraction relating to observations with the material under test, the second

84 cm., $p=38$ cm., $q+2cp=158$ sq. cm., $h=\cdot0000969$.

$\frac{(v_c+v_{-c})^{5/4}}{2}$.	ap sq. cm.	E.	F.	Fh.	$\frac{v_c-v_{-c}}{F.h}$.	Thermal resistance.	Thermal conductance.	$\frac{2a}{q}$.	k.
231·2	·057	13	38100	3·692	0·054				
224·0	6·38	1429	35480	3·432	1·340	1·286	0·789	·00297	·00232
228·9	6·00	1383	36160	3·504	1·341	1·287	0·788	·00281	·00219
226·7	6·35	1439	36610	3·547	0·606	0·552	1·82	·00296	·00539

the observation of three temperatures, as in the original apparatus, and that the additional numerical work required to apply the corrections introduced by the more exact theory of the present paper is not prohibitive.

XC. *Interferometer Measurements in the Extreme Ultra-violet Region of Copper. By* Prof. J. C. McLennan, *F.R.S., and* Miss F. M. Quinlan, *Ph.D.**

IN a recent paper † the authors published a list of wave-lengths of lines in the extreme ultra-violet region of the copper spectrum, λ 2300 Å. to λ 2085 Å., the values being obtained by means of a Fabry and Perot etalon interferometer using standards of longer wave-length as reference lines. This work was carried out in accordance with the recommendation of the International Astronomical Union ‡, so that the region

* Communicated by the Authors.
† Trans. Roy. Soc. Can. 3, vol. xxvi. p. 19 (1932).
‡ Transactions International Astronomical Union, iii. (1928).

of the spectrum below λ 3300 Å. should be studied by interference methods for the purpose of finding radiations which would be suitable to serve as standards of wave-length. The aim of the present research was to extend this table of wave-lengths below λ 2100 Å.

Apparatus and Procedure.

The apparatus used in the earlier investigation was a Fabry and Perot etalon interferometer combined with a Hilger E_1 quartz spectrograph. In order to photograph lines below λ 2100 Å. it was found necessary to use very long exposures, a feature which was not considered desirable owing to the possibility of change in the external conditions during the exposure. In the present research, therefore, an attempt was made to find a dispersing apparatus with a higher light power. This was found in a special spectrograph recently designed by Hilger for work on the Raman Effect. The interference plates were of quartz covered with a thin layer of nickel, and they were separated by fused quartz etalons 1·25 and 2·5 mm. thickness. They were adjusted for parallelism before each set of observations, the light from a mercury arc-lamp being used for this purpose.

Light from the source passed through a quartz astigmatic lens to the interferometer, and the rings formed were focussed on the slit of the spectrograph by means of a quartz-fluorite achromatic lens of 20 cm. focal length. Great care was necessary in aligning the source, the lens, the interferometer, the objective, and the spectrograph.

The reference lines used in making the calculations were the copper lines λ 2135·973 Å., λ 2218·107 Å., λ 2246·998 Å., recently measured by us, and they were obtained from a standard Pfund copper arc. The cathode was a copper rod, 6·1 mm. in diameter, fitted with a cooling cylinder of copper, while the anode, placed below, consisted of a bead of copper oxide on the top of a copper rod 12·3 mm. in diameter. The arc was connected to a source of potential difference of 220 volts, and a current of about 3·7 amperes was used.

The gap between the electrodes was 12 mm. wide, and the light was taken from a section at the centre 1·5 mm. wide, by placing a diaphragm at the position of the image of the arc formed by a condensing lens. For the region below λ 2085 Å., however, it was found that the above arc gave very weak lines, and therefore an arc of the same dimensions as above, but using a 9 mm. gap, was substituted. This was placed a short distance from the astigmatic lens,

and light from the whole arc was allowed to fall on the etalon. The comparison spectrum of the standard Pfund copper arc was photographed before and after that of the copper arc, in order to correct for any variation in temperature and pressure during the exposure, and the average value of the etalon thickness found from these measurements was used in making the calculations. For photographing the region above λ 2100 Å. Ilford Special Rapid Panchromatic plates were found very satisfactory, while below λ 2100 Å, Hilger Schumann plates were used, the time of exposure being less than five minutes for the lines of longer wave-length, while for those below λ 2025 Å. about one hour was required.

Theory *.

When monochromatic light of wave-length λ is incident on an air etalon at an angle θ, the condition for maximum intensity of the transmitted light is given by

$$n\lambda = 2t\cos\theta,$$

where n is a constant, known as the order of interference, and t is the thickness of the etalon.

For normal incidence

$$n_0\lambda = 2t, \text{ where } n_0 \text{ may not be integral.}$$

If, then, the thickness of the etalon be determined by the use of a standard radiation of wave-length λ from the equation

$$n_0\lambda = 2t,$$

then an unknown wave-length λ′ can be calculated from the equation

$$\lambda' = \frac{2t}{n_0'},$$

if its order of interference n_0' is known.

To find n_0' let

$$n_0' = n_1' + \epsilon,$$

where n_1' is the order of interference for the first ring, and ϵ is the fractional part of the order of interference at the centre of the ring system.

It can easily be shown that

$$\epsilon = \frac{(p-1)d_q^2 - (q-1)d_p^2}{d_p^2 - d_q^2},$$

* Fabry and Buisson, Ast. Journ. xxviii. p. 188 (1908). Childs, Journ. Sci. Inst. iii. no. 4, p. 5 (1926).

where d_p and d_q are the linear diameters of the pth and qth rings respectively. The diameters of the rings were obtained with a Hilger photomeasuring micrometer, and calculations of ϵ made from measurements of the first three rings except in cases where the first ring was too diffuse, the second, third, and fourth rings being then used. The value of n_1' was obtained from the equation

$$n_1' = \frac{2t}{\lambda'},$$

an approximate value of λ' being sufficient.

The thickness of the etalon * was measured in terms of copper lines listed above, based on an approximate value obtained by direct measurement.

Corrections.

1. *Phase change.*—It is well known that the penetration of light into a reflecting film varies with different wave-lengths, and a correction must, therefore, be made in the above calculation. The usual method of applying this correction is to use two etalons whose thickness t_1 and t_2 are known for the wave-length λ from the equation

$$2t_1 = n_{01}\lambda,$$
$$2t_2 = n_{02}\lambda.$$

If α is the error in t for another wave-length λ', then

$$2(t_1+\alpha) = n_{01}'\lambda',$$
$$2(t_2+\alpha) = n_{02}'\lambda',$$

whence

$$\lambda' = \frac{2t_1 - 2t_2}{n_{01}' - n_{02}'}.$$

2. *Temperature, pressure, and relative humidity.*—The values of the standards of wave-length have been defined under the conditions of dry air at 15° C. and 760 mm. pressure. Variations in these conditions may cause the absolute values of the wave-lengths to vary by considerable amounts due to a change in the dispersion of the air. A table of the corrections to be applied over a wide range of wave-lengths and densities of air in which spectroscopic measurements are ordinarily made, has been prepared and published by the Bureau of Standards †. In the present research, however, the average temperature and pressure

* Trans. Roy. Soc. Can. 3, vol. xxvi. p. 19 (1932).

† Meggers and Peters, Bull. Bur. Standards, 14, iv. p. 697 (1919).

were 22° C. and 759 mm. respectively, involving a correction to the wave-lengths of about ·001 Å., which was considered negligible. The relative humidity at the time of the experiment was about 30 per cent., but no account was taken of this in the calculations.

Results.

On the photographs of the comparison spectrum of the standard Pfund copper arc were found some lines which had not previously been measured by us. Table I. contains the wave-lengths of these lines measured in terms of the copper lines listed above. Since these lines were near the unknown lines, the error due to phase change was considered very

Table I.

I. Hasbach.	II. Mitra.	III. Shenstone.	IV. McLennan & Quinlan.	N.
λ Å.	λ Å.	λ Å.	λ Å.	
2276·244	2276·261	2276·261	2276·255	2
2189·599	2189·631	2189·631	2189·625	2
2179·37	—	2179·41	2179·401	2
2148·93	2148·897	2148·98	2148·980	2
2130·700	—	2130·76	2130·754	2
2125·978	2126·047	2156·047	2126·035	2
2122·916	—	2122·98	2122·973	2
2112·023	2112·105	2112·09	2112·099	1
2104·717	—	2104·81	2104·771	1

small, and was neglected. This table also contains the values of these wave-lengths as found by other observers *, a concave grating being used to give those in column I., while the ones in column II. were obtained with an interferometer. In column III. is given a list of values used in classifying the copper spectrum which were obtained by correcting Hasbach's values to agree with Mitra's results.

Table II. contains a set of values of wave-lengths of the lines below λ 2100 Å. found by comparing them with the same copper lines as standards, and corrected for phase change by the use of the etalons as already described. In both tables, N, is the number of plates measured.

* Hasbach, *ZS. f. wiss. Phot.* xiii.–xiv. p. 399 (1914). Mitra, *Ann. d. Phys.* xix. p. 315 (1923). Shenstone, Phys. Rev. xxviii. p. 449 (1926); xxix. p. 380 (1927).

Discussion.

The values of the wave-lengths given in the last three columns of Table I. agree very well, except for the line λ 2148 Å., in which case our value agrees with the one used by Shenstone in his calculations. The possible error of these results is rather large, due to the small magnification of the spectrograph used. If the average deviation of the individual readings from the mean reading of the diameters of the rings be found, and its effect on the value of the wave-length deduced, an error of ·01 Å. is possible. This error will increase towards the shorter wave-length, since the diameter of the rings gets smaller. The average devi-

Table II.

I. Hasbach.	II. Shenstone.	III. McLennan & Quinlan.		
λ Å.	λ Å.	λ Å.		
2085·22	2085·30	2085·292	2	A
2054·88	2054·99	2054·952	6	A
2043·71	2043·81	2043·782	3	A
2037·01	2037·13	2037·089	4	A
2035·74	2035·85	2035·82	4	B
	Spark Source.			
2025·36	2025·50	2025·46	4	B
	λ (Vac.)			
1999·55	2000·348	2000·27	4	B
1989·06	1989·860	1989·85	2	C
1979·16	1979 971	1979·95	2	C

ation of the individual from the mean value of the wave-lengths found for different plates is, however, less than this value.

In Table II. the values of the wave-lengths have been divided into classes A, B, and C, the average deviation of the individual values being less than ·01 Å. for A class, less than ·03 Å. for B class, and weak lines are listed in class C. The values in columns II. and III. of Table II. do not show as good an agreement as those of Table I., a possible source of error being the pole effect, since, owing to the difficulty of photographing this region, the arc could not be used under standard conditions. However, the agreement between the individual values of the wave-lengths of some of the lines indicates the possibility of reproducing them.

In conclusion, it would appear that, in the choice of radiations to serve as standards of wave-length in the extreme ultra-violet region of the spectrum, the copper spectrum may provide many lines suitable for this purpose.

The McLennan Laboratory,
University of Toronto.
August 1932.

XCI. *Applications of Prolate Spheroidal Harmonics.* By D. M. Wrinch, *M.A., D.Sc.(Oxon.), Hestha Ayrton Fellow, Girton College, Cambridge, and Suzette Taylor Fellow, Lady Margaret Hall, Oxford* *.

§ 1. Introduction.

THIS paper is concerned with various problems of electrostatics and hydrodynamics associated with prolate spheroids, in cases where there is symmetry about the axis of figure.

The analysis is carried out in terms of prolate spheroidal coordinates (μ, ν) which are related to the cylindrical coordinates (z, ρ) by the equations

$$z = a\mu\nu, \qquad \rho = a(1-\mu^2)^{1/2}(\nu^2-1)^{1/2},$$

where $\mu^2 \leq 1$, $\nu^2 \geq 1$. From the adoption of harmonics in the standard form given by Lamb,

$$\phi = \sum_0^\infty a_n \mathrm{P}_n(\mu)(\mathrm{Q}_n(\nu), \mathrm{P}_n(\nu)),$$

suitable respectively for the space outside and inside a specific spheroid $\nu=\nu_1$, there result various expansions of functions of ϕ in series of terms involving products of $\mathrm{P}_n(\mu)$ and $\mathrm{Q}_n(\nu)$ or $\mathrm{P}_n(\nu)$. The introduction of Stokes's Stream Function ψ also yields expansions of interest, in which the individual terms involve products of $d\mathrm{P}_n(\mu)/d\mu$ and $d\mathrm{Q}_n(\nu)/d\nu$ or $d\mathrm{P}_n(\nu)/d\nu$. As examples, the expansion of the $1/\mathrm{R}_2$ and $1/\mathrm{R}_2^3$, where R_2 denotes the distance between two points whose spheroidal coordinates are $(-1, \nu_2)$ and (μ, ν), may be cited.

Among the problems discussed, that of a prolate spheroidal conductor earthed in the presence of an external electric charge at a point on the axis of figure, may be mentioned; also the problem of a spheroidal obstacle in the presence of a source of fluid similarly placed.

* Communicated by the Author.

§ 2. On Prolate Spheroidal Coordinates.

Prolate spheroidal coordinates (μ, ν) are defined in terms of cylindrical coordinates (z, ρ) by means of the equations

$$z = a\mu\nu, \quad \rho = a(1-\mu^2)^{1/2}(\nu^2-1)^{1/2},$$

where μ takes values from -1 to $+1$, and ν^2 takes values from 1 to ∞. The μ-levels are then the confocal hyperboloids

$$z^2/a^2\mu^2 - \rho^2/a^2(1-\mu^2) = 1,$$

and the ν-levels the confocal prolate spheroids

$$z^2/a^2\nu^2 + \rho^2/a^2(\nu^2-1) = 1.$$

We may notice also that the equation of the prolate spheroids may be written in the form

$$\rho^2 = (\nu^2-1)(a^2-z^2/\nu^2) = (\nu^2-1)z_1z_2/\nu^2,$$

where z_1 and z_2 are the distances along the axis from the two extremities of the major axis of the generating ellipse.

The prolate spheroids range from the rod where $\nu^2=1$ to the sphere at infinity where $\nu=\infty$. The third of the orthogonal curvilinear coordinates associated with μ and ν is ω the azimuthal angle ; the ω-levels are planes.

It is relevant to recall that p the perpendicular distance from the centre on to the tangent plane to the spheroid ν at any point and c the curvature of the generating ellipse are given in (μ, ν) coordinates in the form

$$\frac{1}{c} = \frac{a^4\nu^2(\nu^2-1)}{p^2} = a\nu(\nu^2-1)\left[\frac{\nu^2-\mu^2}{\nu^2(\nu^2-1)}\right]^{3/2},$$

and to notice that δn_ν, the space element normal to the spheroid ν at any point μ, is given by

$$\delta n_\nu = \delta\mu/h_\mu = a\delta\mu(\nu^2-\mu^2)^{1/2}/(1-\mu^2)^{1/2}.$$

§ 3. Prolate Spheroidal Harmonics.

Following Lamb *, we may adopt as the standard harmonics symmetric about z, the axis of figure,

$$\phi = \sum_0^\infty a_n \mathrm{P}_n(\mu)\mathrm{Q}_n(\nu),$$

for the space between a spheroid $\nu=\nu_1>1$ and the sphere at infinity $\nu=\infty$, and

$$\phi = \sum_0^\infty a_n \mathrm{P}_n(\mu)\mathrm{P}_n(\nu)$$

* 'Hydrodynamics,' chap. 5.

for the space inside the spheroid $\nu=\nu_1 \geq 1$, defining the Legendre functions by the equations

$$\begin{aligned} P_n(\mu) &= \frac{1}{2^n n!}\frac{d^n}{d\mu^n}(\mu^2-1)^n \\ &= \frac{1\cdot 3\ldots n-1}{n!}\left[\mu^n - \frac{n(n-1)}{2(2n-1)}\mu^{n-2}\right. \\ &\qquad \left. + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}\mu^{n-4}\ldots\right], \quad (3.1) \end{aligned}$$

for $\mu^2<1$, and

$$\begin{aligned} Q_n(\nu) &= P_n(\nu)\int_\nu^\infty d\zeta/P_n^2(\zeta).(\zeta^2-1) \\ &= \tfrac{1}{2}P_n(\nu)\log\left|\frac{\nu+1}{\nu-1}\right| - \frac{2n-1}{1.n}P_{n-1}(\nu) \\ &\qquad - \frac{2n-5}{3(n-1)}P_{n-3}(\nu)+\ldots \\ &= \frac{n!}{1.3\ldots 2n+1}\left\{\nu^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}\nu^{-n-3}\right. \\ &\qquad \left. + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)}\nu^{-n-5}+\ldots\right\}, \quad (3.2) \end{aligned}$$

for $\nu^2>1$. We may also record the facts that

$$\begin{aligned} Q_0(\nu) &= \tfrac{1}{2}\log|\nu+1| - \tfrac{1}{2}\log|\nu-1|, \\ Q_1(\nu) &= \tfrac{1}{2}\nu[\log|\nu+1| - \log|\nu-1|] - 1, \\ Q_2(\nu) &= \tfrac{1}{2}P_2(\nu)[\log|\nu+1| - \log|\nu-1|] - \tfrac{3}{2}\nu; \end{aligned}$$

and

$$\begin{aligned} Q_0'(\nu) &= -1/(\nu^2-1), \\ Q_1'(\nu) &= -\nu/(\nu^2-1) + \tfrac{1}{2}\log|\nu+1| - \tfrac{1}{2}\log|\nu-1|. \end{aligned}$$

Further,

$$P_n(1) = 1, \qquad P_n'(1) = \tfrac{1}{2}n(n+1).$$

$$\begin{aligned} P_{2m+1}(0) &= 0, \\ P_{2m}(0) &= (-)^m 2m!/[2^{2m}(m!)^2]. \end{aligned}$$

In view of the fact that

$$P_n(\mu)Q_n'(\mu) - Q_n(\mu)P_n'(\mu) = 1/(1-\mu^2)$$

[since $P_n(\mu)$ and $Q_n(\mu)$ share the linear differential equation of the second order

$$\left[\frac{d}{d\mu}(1-\mu^2)\frac{d}{d\mu}+n(n+1)\right]\zeta=0\Big]$$

it follows that

$$\lim_{r=1}\frac{Q_n(\nu)}{Q_m(\nu)}=\frac{P_n(1)}{P_m(1)}=\lim_{r=1}\frac{Q_n'(\nu)}{Q_m'(\nu)}=1.$$

§ 4. Expansions for the Inverse Distance.

Suppose A is any point in space and that its spheroidal coordinates are μ, ν (where $|\mu|<1$, $1\leq\nu$), so that its (z, ρ) coordinates are

$$z=a\mu\nu, \quad \rho=a(1-\mu^2)^{1/2}(\nu^2-1)^{1/2}. \quad . \quad . \quad (4.1)$$

Let B_2 be a point on the axis of symmetry whose distance along the axis is

$$z=-z_2=-a\nu_2<0:$$

and if ν_2 is greater than unity, B will lie outside the line joining the foci F, G, and if ν_2 is less than unity it will lie between the foci: it will always be on the negative part of the z-axis. In particular, when ν_2 is zero, B will be at the centre of the confocal spheroids, and, when it is unity, at the focus F.

It will frequently be necessary to make use of the expansion in P_n and Q_n functions for $1/R_2$, the reciprocal of the distance between (μ, ν) and $(-1, \nu_2)$. Given by Heine* in 1874, it takes the forms

$$\frac{a}{R_2}=\Sigma(-)^n(2n+1)P_n(\mu)[P_n(\nu_2)Q_n(\nu)] \quad . \quad (4.2)$$

when $\nu>\nu_2$, and the form

$$\frac{a}{R_2}=\Sigma(-)^n(2n+1)P_n(\mu)[Q_n(\nu_2)P_n(\nu)] \quad . \quad (4.3)$$

when $\nu<\nu_2$.

It is interesting to notice that expressions of the form

$$\frac{a}{R_2}=\Sigma a_nP_n(\mu)[Q_n(\nu), P_n(\nu)], \quad \nu\gtrless\nu_2,$$

evidently exist, since a/R_2 is the potential of a point-charge at the point $(-1, \nu_2)$. Remarking the fact that there is a

* Heine, 'Kugelfunctionen.'

simple way of reducing prolate spheroidal coordinates to spheroidal polar coordinates (μ, ζ), defined by

$$z = k\mu\zeta, \qquad \rho = k\zeta(1-\mu^2)^{1/2}$$

[so that $k\zeta = r$, $\mu = \cos\theta$] in the degenerate case, by writing

$$a\nu = k\zeta, \qquad a\nu_2 = k\zeta_2,$$

and making a tend to zero, we can at once see the form taken by the coefficients a_n in the inverse distance formula for prolate harmonics. For, from the well-known formulæ for spherical polar coordinates,

$$\frac{1}{R_2} = \frac{1}{(k[\mu\zeta+\zeta_2]^2+k(1-\mu^2)\zeta^2)^{1/2}} = \frac{1}{k[\zeta_2^2+\zeta^2+2\mu\zeta\zeta_2]^{1/2}}$$

$$= \frac{1}{k\zeta_2}\sum_0^\infty (-)^n\left(\frac{\zeta}{\zeta_2}\right)^n P_n(\mu), \qquad \frac{1}{k\zeta}\sum_0^\infty (-)^n\left(\frac{\zeta_2}{\zeta}\right)^n P_n(\mu),$$

according as $\zeta \lesseqgtr \zeta_2$, we have

$$a_n = c_n[P_n(\zeta),\ Q_n(\zeta_2)], \qquad \zeta \gtreqless \zeta_2,$$

where

$$c_n \lim_{a=0} \frac{1}{a} P_n\left(\frac{k\zeta}{a}\right) Q_n\left(\frac{k\zeta_2}{a}\right) = \frac{1}{k\zeta_2}(-)^n\left(\frac{\zeta}{\zeta_2}\right)^n, \quad \zeta < \zeta_2,$$

$$c_n \lim_{a=0} \frac{1}{a} Q_n\left(\frac{k\zeta}{a}\right) P_n\left(\frac{k\zeta_2}{a}\right) = \frac{1}{k\zeta}(-)^n\left(\frac{\zeta_2}{\zeta}\right)^n, \quad \zeta > \zeta_2.$$

But by (3.1)

$$\lim_{a=0} a^n P_n(k\zeta_2/a) = \frac{2n!}{2^n n!\, n!}(k\zeta_2)^n$$

and since $k\zeta/a > 1$, by (3.2)

$$\lim Q_n(k\zeta/a)a^{-n-1} = \frac{n!\, n!\, 2^n}{(2n+1)!}(k\zeta)^{-n-1}.$$

Therefore when $\zeta > \zeta_2$,

$$c_n = (-)^n(2n+1).$$

Thus when $\zeta > \zeta_2$,

$$a_n = (-)^n(2n+1)P_n(\zeta_2).$$

Similarly when $\zeta < \zeta_2$,

$$a_n = (-)^n(2n+1)Q_n(\zeta_2).$$

The inverse distance formula of Heine then results.

§ 5. Relations between ϕ and ψ-Functions.

We are concerned only with harmonics which are symmetrical about the axis of z. We can therefore introduce a function ψ associated with each potential function ϕ which is symmetrical about the axis of z—called in hydrodynamical applications Stokes's Stream Function—by means of the definition

$$\partial\psi/\partial z = -\rho\,\partial\phi/\partial\rho,$$
$$\partial\psi/\partial\rho = +\rho\,\partial\phi/\partial z,$$

which specify ψ apart from an arbitrary constant. The equation for ψ in (z, ρ) coordinates is evidently

$$\frac{\partial^2\psi}{\partial z^2} + \rho\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}\right) = 0,$$

and we can easily establish the correspondence of

$$\phi = \frac{\partial^n}{\partial z^n}\left[\rho^2 + (z+z_2)^2\right]^{-1/2} = \frac{\partial^n}{\partial z^n}\left(\frac{1}{\mathrm{R}_2}\right)$$

and

$$\psi + \text{constant} = \frac{\partial^{n+1}}{\partial z^{n+1}}\left[\rho^2 + (z+z_2)^2\right]^{1/2} = \frac{\partial^{n+1}\mathrm{R}_2}{\partial z^{n+1}}$$

in the usual manner *, and also the correspondence of

$$\phi = \int\frac{1}{\mathrm{R}_2}\,dz = \sinh^{-1}\left(\frac{z+z_2}{\rho}\right)$$

and

$$\psi + \text{constant} = \left[\rho^2 + (z+z^2)^2\right]^{1/2} = \mathrm{R}_2,$$

covered by the form given above if we use the interpretation

$$\frac{\partial^{-1}}{\partial z^{-1}}f(z) = \int(z)\,dz.$$

Now, for any orthogonal curvilinear coordinates (μ, ν, ω), where ω is the azimuthal angle, the function ψ corresponding to ϕ may be defined—apart from an arbitrary constant—by

$$h_\mu\,\partial\psi/\partial\mu = -\rho h_\nu\,\partial\phi/\partial\nu,$$
$$h_\nu\,\partial\psi/\partial\nu = +\rho h_\mu\,\partial\phi/\partial\mu,$$

* Lamb, *loc. cit.* pp. 120–121.

which gives in the case of the prolate spheroidal coordinates used in this paper,

$$\partial\psi/\partial\mu = -a(\nu^2-1)\,\partial\phi/\partial\nu,$$
$$\partial\psi/\partial\nu = +a(1-\mu^2)\,\partial\phi/\partial u.$$

So that the equation for ψ is

$$(1-\mu^2)\,\partial^2\psi/\partial\mu^2 + (\nu^2-1)\,\partial^2\psi/\partial\nu^2 = 0$$

or

$$\left[(1-\mu^2)\frac{\partial^2}{\partial\mu^2} + n(n+1)\right]\psi + \left[(\nu^2-1)\frac{\partial^2}{\partial\nu^2} - n(n+1)\right]\psi = 0.$$

But

$$\left[\frac{d}{d\zeta}(1-\zeta^2)\frac{d}{d\zeta} + n(n+1)\right](\mathrm{P}_n(\zeta),\ \mathrm{Q}_n(\zeta)) = 0,$$

and therefore

$$\left[\frac{d^2}{d\zeta^2}(1-\zeta^2) + n(n+1)\right]\left(\frac{d}{d\zeta}\mathrm{P}_n(\zeta),\ \frac{d}{d\zeta}\mathrm{Q}_n(\zeta)\right) = 0.$$

So that the equation

$$(1-\zeta^2)\frac{d^2y}{d\zeta^2} + n(n+1)y = 0$$

is satisfied by $(1-\zeta^2)\mathrm{P}_n'(\zeta)$ and by $(1-\zeta^2)\mathrm{Q}_n'(\zeta)$. Hence solutions for ψ may be put in the form

$$\psi = \sum_0^\infty [\mathrm{A}_n(1-\mu^2)\mathrm{P}_n'(\mu),\ \mathrm{B}_n(1-\mu^2)\mathrm{Q}_n'(\mu)] \cdot [\mathrm{C}_n(\nu^2-1)\mathrm{P}_n'(\nu),\ \mathrm{D}_n(\nu^2-1)\mathrm{Q}_n'(\nu)]. \quad (5.1)$$

Moreover, from (6) and the fact that

$$\left[\frac{d}{d\zeta}(1-\zeta^2)\frac{d}{d\zeta} + n(n+1)\right](\mathrm{P}_n(\zeta),\ \mathrm{Q}_n(\zeta)) = 0,$$

we can deduce the correspondence of the general form adopted for ϕ, namely,

$$\phi = \mathrm{P}_n(\mu)[b_n.\mathrm{P}_n(\nu) + c_n\mathrm{Q}_n(\nu)]$$

and

$$\psi + \text{constant} = \frac{a}{n(n+1)}(1-\mu^2)\mathrm{P}_n'(\mu) \cdot [b_n(\nu^2-1)\mathrm{P}_n'(\nu) + c_n(\nu^2-1)\mathrm{Q}_n'(\nu)]. \quad (5.2)$$

This result is given by Lamb * for $n \neq 0$. But it is evidently

* *Loc. cit.* p. 134.

true also for $n=0$ if we correctly interpret the symbol $P_n'(\mu)/n$ for the value $n=0$. For, using Murphy's definition * of the function $P_n(\zeta)$ as a hypergeometric function of $(\frac{1}{2}-\frac{1}{2}\zeta)$ in the form

$$P_n(\zeta) = F(n+1, -n\,;\ 1\,;\ \tfrac{1}{2}-\tfrac{1}{2}\zeta),$$

when $|1-\zeta| \leq 2$, we have, since $|\mu| \leq 1$,

$$\begin{aligned}
&\operatorname*{lt}_{n\to 0} (P_n'(\mu)/n) \\
&\quad = \operatorname*{lt}_{n\to 0} \frac{1}{n}\frac{d}{d\mu} F(1, -n\,;\ 1\,;\ \tfrac{1}{2}-\tfrac{1}{2}\mu) \\
&\quad = \operatorname*{lt}_{n\to 0} \frac{1}{n}\frac{d}{d\mu}\left\{1-\frac{n}{1}(\tfrac{1}{2}-\tfrac{1}{2}\mu)-\frac{n}{2}(\tfrac{1}{2}-\tfrac{1}{2}\mu)^2-\frac{n}{3}(\tfrac{1}{2}-\tfrac{1}{2}\mu)^3\ldots\right\} \\
&\quad = \tfrac{1}{2}[1+(\tfrac{1}{2}-\tfrac{1}{2}\mu)+(\tfrac{1}{2}-\tfrac{1}{2}\mu)^2+\ \ldots] \\
&\quad = 1/(1+\mu). \qquad\qquad (5.3)
\end{aligned}$$

Thus, corresponding to

$$\begin{aligned}
\phi &= P_0(\mu)[b_0 P_0(\nu)+c_0 Q_0(\nu)] \\
&= b_0+\tfrac{1}{2}c_0 \log\frac{\nu+1}{\nu-1},
\end{aligned}$$

we have

$$\begin{aligned}
\psi+\text{constant} &= a(1-\mu^2)[b_0(\nu^2-1)P_0'(\nu)+c_0(\nu^2-1)Q_0'(\nu)] \\
&= 0+ac_0(\mu-1).
\end{aligned}$$

§ 6. Stokes's Stream Function.

In hydrodynamical applications the expansions of various ϕ and ψ functions are required. The fundamental ϕ-expansion, namely,

$$\begin{aligned}
\phi = a/R_2 &= [(\mu\nu+\nu_2)^2+(1-\mu^2)(\nu^2-1)]^{-1/2} \\
&= \sum_0^\infty (-)^n(2n+1)P_n(\mu)[P_n(\nu_2)Q_n(\nu),\ Q_n(\nu_2)P_n(\nu)], \\
&\qquad\qquad \ldots \qquad (6.1)
\end{aligned}$$

for $\nu > \nu_2$ and $\nu < \nu_2$ respectively, has already been given. We now proceed to find the corresponding ψ-function, namely,

$$\psi = a\frac{\partial}{\partial z}(R_2) = \frac{a(z+z_2)}{R_2} = \frac{a(\mu\nu+\nu_2)}{[(\mu\nu+\nu_2)^2+(1-\mu^2)(\nu^2-1)]^{1/2}}.$$

* See Whittaker and Watson, 'Modern Analysis,' p. 311 (1920).

This expansion may at once be written down—except for a constant which will be determined later—since it must be the ψ-expansion corresponding to ϕ-expansion given in (6·1). We have, in fact,

$$\psi = a(\mu\nu+\nu_2)/[(\mu\nu+\nu_2)^2+(1-\mu^2)(\nu^2-1)]^{1/2}$$

$$= A+a(1-\mu^2)(\nu^2-1)\sum_0^\infty (-)^n \frac{2n+1}{n(n+1)} P_n'(\mu)P_n(\nu_2)Q_n'(\nu), \quad \nu>\nu_2,$$

$$= B+a(1-\mu^2)(\nu^2-1)\sum_0^\infty (-)^n \frac{2n+1}{n(n+1)} P_n'(\mu)Q_n(\nu_2)P_n'(\nu), \quad \nu<\nu_2,$$

merely by writing down the corresponding ψ-terms one by one by means of (5.2). To determine A we remark that the limit of ψ as $\nu\to\infty$ is $a\mu$, and therefore

$$A-a(1-\mu) = a\mu$$

(since $(\nu^2-1)Q_0'(\nu)=-1$) and in consequence $A=a$. To determine B we remark that the limit of ψ as $\nu_2\to\infty$ is a, and therefore $B=a$ also. Thus we have the important ψ-expansion—meaning by a ψ-expansion, an expansion of the type given in (5.1)—

$$\frac{\mu\nu+\nu_2}{[(\mu\nu+\nu_2)^2+(1-\mu^2)(\nu^2-1)]^{1/2}}$$

$$= 1+(1-\mu^2)(\nu^2-1)\sum_0^\infty (-)^n \frac{2n+1}{n(n+1)} P_n'(\mu)P_n(\nu_2)Q_n'(\nu), \quad \nu>\nu_2, \quad (6.2)$$

$$= 1+(1-\mu^2)(\nu^2-1)\sum_0^\infty (-)^n \frac{2n+1}{n(n+1)} P_n'(\mu)Q_n(\nu_2)P_n'(\nu), \quad \nu<\nu_2, \quad (6.3)$$

or if we interpret the symbol $P_n'(\mu)/n$, when $n=0$, as in (5.3), the expansion stands in the form

$$\frac{\mu\nu+\nu_2}{[(\mu\nu+\nu_2)^2+(1-\mu^2)(\nu^2-1)]^{1/2}}$$

$$= \mu+(1-\mu^2)(\nu^2-1)\sum_1^\infty (-)^n \frac{2n+1}{n(n+1)} P_n'(\mu)P_n(\nu_2)Q_n'(\nu), \quad \nu>\nu_2, \quad (6.4)$$

$$= 1+(1-\mu^2)(\nu^2-1)\sum_1^\infty (-)^n \frac{2n+1}{n(n+1)} P_n'(\mu)Q_n(\nu_2)P_n'(\nu), \quad \nu<\nu_2, \quad (6.5)$$

There are interesting particular cases of the formulæ (6.4) and (6.5). First, if $\nu_2=1<\nu$,

$$\frac{\mu\nu+1}{\mu+\nu}=\mu+(1-\mu^2)(\nu^2-1)\sum_1^\infty(-)^n\frac{2n+1}{n(n+1)}\mathrm{P}_n'(\mu)\mathrm{P}_n(1)\mathrm{Q}_n'(\nu).$$

But

$$\mathrm{P}_n(1)=1.$$

Therefore

$$\frac{\mu\nu+1}{\mu+\nu}=\mu+(1-\mu^2)(\nu^2-1)\sum_1^\infty(-)^n\frac{2n+1}{n(n+1)}\mathrm{P}_n'(\mu)\mathrm{Q}_n'(\nu).$$

Or if $\mu\neq1$,

$$1/(\mu+\nu)=(\nu^2-1)\sum_1^\infty(-)^n\frac{2n+1}{n(n+1)}\mathrm{P}_n'(\mu)\mathrm{Q}_n'(\nu).$$

Again, if $\nu_2=0$, $\nu>1$,

$$\frac{\mu\nu}{(\mu^2+\nu^2-1)^{1/2}}$$

$$=\mu+(1-\mu^2)(\nu^2-1)\sum_1^\infty(-)^n\frac{2n+1}{n(n+1)}\mathrm{P}_n'(\mu)\mathrm{P}_n(\nu)\mathrm{Q}_n'(\nu),$$

But

$$\mathrm{P}_{2m+1}(0)=0,\qquad\mathrm{P}_{2m}(0)=(-)^m\frac{2m!}{2^{2m}m!\,m!}.$$

Therefore, if $\nu>1$,

$$\frac{\mu\nu}{(\mu^2+\nu^2-1)^{1/2}}=\mu+(1-\mu^2)(\nu^2-1)\sum_1^\infty(-)^m\frac{2m!}{2^{2m}m!\,m!}$$

$$\times\frac{4m+1}{2m(2m+1)}\mathrm{P}_{2m}'(\mu)\mathrm{Q}_{2m}'(\nu).$$

Further, if $\mu=0$, we have

$$\frac{2\nu}{(\nu_2^2+\nu^2-1)^{1/2}}$$

$$=(\nu^2-1)\sum_1^\infty(-)^n\frac{2n+1}{n(n+1)}\mathrm{P}_n'(0)\mathrm{P}_n(\nu_2)\mathrm{Q}_n'(\nu),\qquad\nu>\nu_2,$$

$$=1+(\nu^2-1)\sum_1^\infty(-)^n\frac{2n+1}{n(n+1)}\mathrm{P}_n(0)\mathrm{Q}_n(\nu_2)\mathrm{P}_n'(\nu),\quad\nu<\nu_2.$$

But

$$\mathrm{P}'_{n+1}(0)=(n+1)\mathrm{P}_n(0).$$

Hence

$$\frac{\nu_2}{(\nu_2^2+\nu^2-1)^{1/2}}$$

$$= (\nu^2-1)\sum_0^\infty (-)^m \frac{(2m+1)!}{2^{2m}m!\,m!} \times \frac{4m+3}{(2m+1)(2m+2)} P_{2m+1}(\nu_2)Q'_{2m+1}(\nu), \quad \nu > \nu_2,$$

$$= 1+(\nu^2-1)\sum_0^\infty (-)^m \frac{(2m+1)!}{2^{2m}m!\,m!} \times \frac{4m+3}{(2m+1)(2m+2)} Q_{2m+1}(\nu_2)P'_{2m+1}(\nu), \quad \nu < \nu_2.$$

Other results may be obtained by inserting other particular values of μ, ν, ν_2.

The fundamental ψ-expansion could evidently have been obtained by the obvious method of multiplying the inverse Distance Formula (6.1) by $(\mu\nu+\nu_2)$ and substituting for terms in $\mu P_n(\mu)$, $\nu P_n(\nu)$, and $\nu Q_n(\nu)$ by means of the recurrence relations

$$(\zeta^2-1)f_n'(\zeta) = n(\zeta f_n(\zeta)-f_{n-1}(\zeta)) = (n+1)(f_{n+1}(\zeta)-\zeta f_n(\zeta)),$$

satisfied both by $f_n(\zeta)=P_n(\zeta)$ and by $f_n(\zeta)=Q_n(\zeta)$. We should then find that expansions result which are in fact of the ψ-type. We have not adopted this method, not because it is laborious—as it is—but because it is far more interesting to recognise *ab initio* the fact that we are dealing with a ψ-function which will necessarily have an expansion of the type given in (5.1) and to use the correspondence, of which the theory of the ψ-function has already demonstrated the existence, between this expansion and the expansion for the inverse distance.

§ 7. Electrostatic Problems.

The Spheroidal Conductor freely charged with Electricity.

We consider a conductor in the form of the spheroid $\nu=\nu_1$. It is freely charged to potential V_1. Then, since its potential must be of the form

$$V = \sum_0^\infty a_n P_n(\mu)Q_n(\nu)$$

and constant on the spheroid $\nu=\nu_1$, evidently

$$V = V_1 Q_0(\nu)/Q_0(\nu_1).$$

The charge E and the capacity C of the conductor are given by

$$C = E/V_1 = \lim_{r=\infty} Vr/V_1.$$

Now

$$r^2 = a^2[\mu^2\nu^2 + (1-\mu^2)(\nu^2-1)]:$$

therefore r and ν tend to infinity together in such a manner that

$$\lim_{r=\infty} r/a\nu = 1.$$

Thus

$$C = E/V_1 = \lim_{\nu=\infty} a\nu V/V_1$$

$$= \frac{a}{Q_0(\nu_1)} \lim_{\nu=\infty} (\nu Q_0\nu) = \frac{a}{Q_0(\nu_1)}.$$

Hence, a prolate spheroid whose generating ellipse has major axis

$$2b = 2a\nu_1$$

and eccentricity

$$e = 1/\nu_1$$

has capacity

$$C = 2be/\log\left(\frac{1+e}{1-e}\right),$$

giving as usual, when e tends to zero, the capacity b for a sphere of radius b.

The surface density σ is given by

$$4\pi\sigma = -\left(h_\nu \frac{\partial V}{\partial \nu}\right)_{\nu=\nu_1}$$

$$= -\frac{V_1}{a}\left(\frac{\nu_1^2-1}{\nu_1^2-\mu^2}\right)^{1/2} \frac{Q_0'(\nu_1)}{Q_0(\nu_1)}$$

$$= \frac{2V_1}{a} \Big/ \left[(\nu_1^2-\mu^2)^{1/2}(\nu_1^2-1)^{1/2} \log\left(\frac{\nu_1+1}{\nu_1-1}\right)\right]$$

$$= 2V_1 pe/[b^2(1-e^2)\log[(1+e)/(1-e)]],$$

in the usual way, p being the perpendicular from the centre of the spheroid on its tangent at the point in question.

The case of the rod of length $2a$, which results when $\nu_1 = 1$, is also of interest. We then have

$$V = \left[\frac{V_0}{Q_0(\nu_1)}\right]_{\nu_1=1} \times Q_0(\nu)$$

and

$$4\pi\sigma = \left[\frac{1}{Q_0(\nu_1)(\nu_1^2-1)^{1/2}}\right]_{\nu_1=1} \times \frac{V_1}{a(1-\mu^2)^{1/2}}$$

$$= \left[\frac{1}{Q_0(\nu_1)(\nu_1^2-1)^{1/2}}\right]_{\nu_1=1} \times \frac{V_1}{(a^2-z^2)^{1/2}}.$$

Hence σ is inversely proportional to the geometrical mean of the distances from the ends. Further, the capacities of various bars, being for a bar of length $2a$, equal to

$$a/[Q_0(\nu_1)]_{\nu_1=1},$$

are directly proportional to their lengths.

The Spheroidal Conductor earthed in a Uniform Field of Electric Force parallel to its Axis of Figure.

In this problem, the undisturbed field has the potential

$$V_0 = fz = faP_1(\mu)P_1(\nu).$$

The potential V_1, thrown off by the earthed conductor $\nu=\nu_1$, is evidently given by

$$V_1 = -faP_1(\mu)P_1(\nu)\,Q_1(\nu)/Q_1(\nu_1),$$

since V_0+V_1 is to be zero on ν_1 and V_1 is evanescent at infinity. The surface density at any point μ on ν_2 is then given by

$$4\pi\sigma = -\left[h_\nu \frac{\partial}{\partial \nu}(V_0+V_1)\right]_{\nu=\nu_1}$$

$$= -f\left(\frac{\nu_1^2-1}{\nu_1^2-\mu^2}\right)^{1/2} P_1(\mu)\left[P_1'(\nu_1) - \frac{P_1(\nu)}{Q_1(\nu_1)}Q_1'(\nu_1)\right]$$

$$= -\frac{f}{(\nu_1^2-1)^{1/2}} \times \frac{\mu}{(\nu_1^2-\mu^2)^{1/2}}$$

$$= -\frac{fe^2}{b^2(1-e^2)Q_1(1/e)} \times pz.$$

Hence, in the particular case of the rod also, σ is proportional to

$$\frac{\mu}{(\nu_1^2-\mu^2)^{1/2}} = \frac{z}{(a^2-z^2)^{1/2}}.$$

These problems are, of course, particular cases of those relating to ellipsoids in general, and the facts that the surface densities are proportional to p and to pz respectively are well

known. The next problem, however, employs a similar procedure in a case where the general results for the ellipsoid have not to be obtained.

The Spheroidal Conductor earthed in the Presence of a Charge at an External Point on its Axis.

If there is a point charge $-\mathrm{E}$ at the external point B_2, given by $\mu=-1$, $\nu=\nu_2$, on the axis of the conductor $\nu=\nu_1$, its undisturbed potential is

$$\mathrm{V}_2=-\frac{\mathrm{E}}{\mathrm{R}_2}.$$

Using the inverse distance formula, this may be written, for values of ν not less and not greater than ν_2, in the respective forms,

$$\mathrm{V}_2=-\frac{\mathrm{E}}{a}\sum_0^\infty(-)^n(2n+1)\mathrm{P}_n(\mu)\left[\mathrm{Q}_n(\nu)\mathrm{P}_n(\nu_2),\ \mathrm{P}_n(\nu)\mathrm{Q}_n(\nu_2)\right].$$

In consequence, the potential thrown off by the conductor, which must be of the form

$$\Sigma a_n\mathrm{P}_n^{\frac{1}{2}}(\mu)\mathrm{Q}_n(\nu),$$

is, in fact,

$$\mathrm{V}_1=\frac{\mathrm{E}}{a}\sum_0^\infty(-)^n(2n+1)\mathrm{P}_n(\mu)\frac{\mathrm{P}_n(\nu_1)\mathrm{Q}_n(\nu_2)}{\mathrm{Q}_n(\nu_1)}\mathrm{Q}_n(\nu),$$

and the whole potential of the field, for $\nu_1<\nu<\nu_2$ is therefore

$$\begin{aligned}\mathrm{V}&=\mathrm{V}_2+\mathrm{V}_1\\&=\frac{\mathrm{E}}{a}\sum_0^\infty(-)^n(2n+1)\mathrm{P}_n^{\frac{1}{2}}(\mu)\mathrm{Q}_n^{\frac{1}{2}}(\nu_2)\left[\frac{\mathrm{P}_n(\nu_1)}{\mathrm{Q}_n(\nu_1)}\mathrm{Q}_n(\nu)-\mathrm{P}_n(\nu)\right].\end{aligned}$$

The total induced charge on the conductor is c, given by

$$\begin{aligned}c&=\lim_{r=\infty}r\mathrm{V}_1=\lim_{\nu=\infty}a\nu\mathrm{V}_1=\mathrm{E}\mathrm{Q}_0(\nu_2)/\mathrm{Q}_0(\nu_1)\\&=\mathrm{E}\log\left(\frac{\nu_2+1}{\nu_2-1}\right)\Big/\log\left(\frac{\nu_1+1}{\nu_1-1}\right).\end{aligned}$$

Thus, also,

$$\begin{aligned}4\pi\sigma&=-\frac{\mathrm{E}}{a^2}\left(\frac{\nu_1^2-1}{\nu_1^2-\mu^2}\right)^{1/2}\sum_0^\infty(-)^n(2n+1)\mathrm{P}_n(\mu)\mathrm{Q}_n(\nu_2)\\&\qquad\times\left[\frac{\mathrm{P}_n(\nu_1)}{\mathrm{Q}_n(\nu_1)}\mathrm{Q}_n'(\nu_1)-\mathrm{P}_n'(\nu_1)\right]\\&=\frac{\mathrm{E}}{a^2(\nu_1^2-\mu^2)^{1/2}(\nu_1^2-1)^{1/2}}\sum_0^\infty(-)^n(2n+1)\frac{\mathrm{P}_n(\mu)\mathrm{Q}_n(\nu_2)}{\mathrm{Q}_n(\nu_1)}.\end{aligned}$$

The semi-major axis of the spheroid is $a\nu_1$, and the distance of the point charge from its centre is $a\nu_2$. This result therefore admits excellent approximations when the charge is far enough away.

The Conducting-rod earthed in the Presence of a Point Charge on its Axis.

The case of the rod—when $\nu = 1$—is also of interest here. For the rod of length $2a$,

$$V_1 = \frac{E}{[aQ_0(\nu_1)]_{\nu_1=1}} \sum_0^\infty (-)^n (2n+1) P_n(\mu) Q_n(\nu_2) Q_n(\nu),$$

so that

$$4\pi\sigma = \frac{1}{[(\nu_1^2-1)^{1/2} Q_0(\nu_1)]_{\nu_1=1}} \times \frac{E}{a^2(1-\mu^2)^{1/2}} \sum_0^\infty (-)^n (2n+1) P_n(\mu) Q_n(\nu_2) \left[\frac{Q_0(\nu_1)}{Q_n(\nu_1)}\right]_{\nu_1=1}$$

$$= \frac{1}{[(\nu_1^2-1)^{1/2} Q_0(\nu_1)]_{\nu_1=1}} \times \frac{E}{a^2(1-\mu^2)^{1/2}} \sum_0^\infty (-)^n (2n+1) P_n(\mu) Q_n(\nu_2) P_n(1)$$

$$= \frac{1}{[(\nu_1^2-1)^{1/2} Q_0(\nu_1)]_{\nu_1=1}} \times \frac{E}{a^2(1-\mu^2)^{1/2}} \times \frac{a}{R_2},$$

where R_2 is the distance from the point charge. Thus σ is proportional to

$$a/(1-\mu^2)^{1/2} R_2 = a^2/[(a^2-z^2)^{1/2}(z+b)] = a^2/(z+b)(z_1 z_2)^{1/2},$$

where z_1 and z_2 are the distances from the two ends of the rod.

It will be remarked that the surface densities at the ends nearer and more remote from the charge, while both infinite, stand to one another in the ratio

$$\frac{1}{b-a} \Big/ \frac{1}{b+a} = \frac{b+a}{b-a},$$

where $b = a\nu_2$ is the distance of the point charge at B_2 from the centre of the rod.

The total induced charge is

$$e = \frac{E}{[Q_0(\nu_1)]_{\nu_1=1}} \times Q_0(\nu_2) = \frac{E}{2[Q_0(\nu_1)]_{\nu_1=1}} \times \log \frac{b+a}{b-a}.$$

Thus when $b/a=2, 3, 4 \ldots$, the induced charges stand in the ratios of $\log 3, \log 2, \log 5/3, \ldots$. Thus, also, the induced charges when rods of length $2a_1, 2a_2, \ldots$, are earthed in the presence of a point charge E at a distance b from their centres along the line of their lengths, are in the ratio

$$\log\frac{b+a_1}{b-a_1} : \log\frac{b+a_2}{b-a_2} \ldots .$$

An Earthed Conductor containing a Spheroidal Cavity under the Influence of a Point Charge at an Internal Point on its Axis.

So far we have considered only the case when $\nu_2 > \nu_1$ and the charge lies outside the spheroidal conductor. But suppose that $\nu_2 < \nu_1$, so that the point charge lies within a spheroidal cavity $\nu = \nu_1$ in an earthed conductor. There will then be a surface density of electrification on the surface of the cavity. Similar analysis immediately gives the potential V_1 in the form

$$V_1 = \frac{E}{a}\sum_0^\infty (-)^n(2n+1)P_n(\mu)\frac{Q_n(\nu_1)P_n(\nu_2)}{P_n(\nu_1)}P_n(\nu), \quad (\nu<\nu_1),$$

and if $\nu_2 < \nu < \nu_1$,

$$V = V_1 + V_2$$

$$= \frac{E}{a}\sum_0^\infty (-)^n(2n+1)P_n(\mu)P_n(\nu_2)\left[\frac{Q_n(\nu_1)}{P_n(\nu_1)}P_n^{\mathrm{I}}(\nu) - Q_n(\nu)\right],$$

so that

$$4\pi\sigma = \frac{E}{a^2}\left(\frac{\nu_1^2-1}{\nu_1^2-\mu^2}\right)^{1/2}\sum_0^\infty (-)^n(2n+1)P_n(\mu)P_n(\nu_2) \times\left[\frac{Q_n(\nu_1)}{P_n(\nu_1)}P_n'(\nu_1) - Q_n'(\nu_1)\right]$$

$$= \frac{E}{a^2(\nu_1^2-\mu^2)^{1/2}(\nu_1^2-1)^{1/2}}\sum_0^\infty (-)^n(2n+1)\frac{P_n(\mu)P_n(\nu_1)}{P_n(\nu_1)}.$$

§ 8. Hydrodynamical Problems.

The solution of the problem of an obstacle in the form of the prolate spheroid ν_1 moving through a fluid at rest at infinity, with a velocity q_0 along its axis, is well known. For the surface condition to be satisfied on $\nu = \nu_1$ is*

$$\psi = \tfrac{1}{2}q_0\rho^2,$$

* H. Lamb, 'Hydrodynamics,' p. 135 (1924).

where ψ is Stokes's stream function for the motion, so that evidently ψ, which is evanescent on the sphere at infinity and must be of the form given in (5.1) is, in fact, given by

$$\psi = \tfrac{1}{2}q_0a^2(1-\mu^2)(\nu^2-1)\mathrm{Q}_1'(\nu)\,\mathrm{P}_1'(\mu)/\mathrm{Q}_1'(\nu_1),$$

and the corresponding ϕ-function is

$$\phi = q_0a\mathrm{Q}(\nu)\,\mathrm{P}_1(\mu)/\mathrm{Q}_1'(\nu).$$

Suppose, now, that the prolate spheroid ν_1 is moving with velocity q_0 along its axis, but contains fluid instead of moving through it. Then the surface condition remains unchanged, but the form adopted for ψ must remain finite within the spheroid. In consequence, we have

$$\begin{aligned}\psi &= \tfrac{1}{2}q_0a^2(1-\mu^2)(\nu^2-1)\mathrm{P}_1'(\nu)\,\mathrm{P}_1'(\mu)/\mathrm{P}_1'(\nu_1)\\ &= \tfrac{1}{2}q_0a^2(1-\mu^2)(\nu^2-1).\end{aligned}$$

The corresponding ϕ-function is

$$\phi = q_0a\mathrm{P}_1(\nu)\,\mathrm{P}_1(\mu)/\mathrm{P}_1'(\nu_1) = q_0a\nu\mu.$$

The Spheroidal Obstacle in the Presence of a Source of Fluid at a Point on its Axis.

The potential of the source of strength m at a point $z=-z_2=-a\nu_2$ on the axis of and external to the spheroid $\nu=\nu_1$, ν_2 being greater than ν_1, is

$$\phi_2 = m/\mathrm{R}_2 = m/a[(\mu\nu+\nu_2)^2+(1-\mu^2)(\nu^2-1)]^{1/2},$$

and its stream function—apart from an arbitrary constant—is

$$\psi_2 = m(z+z_2)/\mathrm{R}_2 = m(\mu\nu+\nu_2)/[(\mu\nu+\nu_2)^2+(1-\mu^2)(\nu^2-1)]^{1/2}.$$

If ψ_1 be the stream function thrown out by the spheroid ν_1, then on the spheroid ν_1, $\psi_1+\psi_2$ must reduce to a constant. ψ_1, moreover, must be evanescent on the sphere at infinity and, in consequence, is of the form

$$\psi_1 = (1-\mu^2)(\nu^2-1)\sum_0^\infty h_n\mathrm{P}_n'(\mu)\mathrm{Q}_n'(\nu).$$

Now, on ν_1, since $\nu_1<\nu_2$,

$$\psi_2 = 1+(1-\mu^2)(\nu^2-1)\sum_0^\infty(-)\,\frac{2n+1}{n(n+1)}\mathrm{P}_n'(\mu)\mathrm{Q}_n(\nu_2)\mathrm{P}_n'(\nu).$$

Hence, putting

$$-b_n Q_n'(\nu_1) = (-)^n \frac{2n+1}{n(n+1)} Q_n(\nu_2) P_n'(\nu_1),$$

we satisfy the surface condition and obtain ψ_1 in the form

$$\psi_1 = -m(1-\mu^2)(\nu^2-1) \sum_0^\infty (-)^n \frac{2n+1}{n(n+1)} P_n'(\mu) \times \frac{Q_n'(\nu)}{Q_n'(\nu_1)} Q_n(\nu_2) P_n'(\nu_1).$$

The corresponding ϕ-function is

$$\phi_1 = -\frac{m}{n} \sum_0^\infty (-)^n (2n+1) P_n(\mu) \frac{Q_n(\nu)}{Q_n'(\nu_1)} Q_n(\nu_2) P_n'(\nu_1).$$

The ϕ-function for the fluid motion is, in consequence,

$$\phi = \phi_1 + \phi_2 = \frac{m}{n} \sum_0^\infty (-)^n (2n+1) P_n(\mu) Q_n(\nu_2) \times \left[P_n(\nu) - \frac{P_n'(\nu_1)}{Q_n'(\nu_1)} Q_n(\nu) \right].$$

In particular, on the spheroid ν_1 itself,

$$\phi = (\phi)_{\nu=\nu_1} = -\frac{m}{n} \sum_0^\infty (-)^n (2n+1) P_n(\mu) \frac{Q_n(\nu_2)}{Q_n'(\nu_1)}.$$

This expansion, convergent when $\nu_1 < \nu_2$, supplies excellent approximations when ν_1/ν_2 is sufficiently small.

Consider, now, q the velocity of the fluid at the surface of the spheroid, which is given by

$$q^2 = (\partial\phi/\partial n_\mu)^2_{\nu=\nu_1} = (h_\mu \partial\phi/\partial\mu)^2_{\nu=\nu_1} = \frac{1-\mu^2}{a^2(\nu^2-\mu^2)} \times \left[\frac{m}{a(\nu_1^2-1)} \sum_0^\infty (-)^n (2n+1) P_n'(\mu) \frac{Q_n(\nu_2)}{Q_n'(\nu_1)} \right]^2,$$

and is required in determining the remittant pressure p on the spheroid. We have, if σ be the density of the fluid, measuring the pressure p towards the source,

$$p = -\tfrac{1}{2}\sigma \int q^2 . 2\pi\rho \, d\rho,$$

taken over the spheroid. Thus

$$p = \frac{\pi\sigma m^2}{(\nu_1^2-1)a^2}\int_{-1}^{+1}\frac{\mu(1-\mu^2)}{\nu_1^2-\mu^2} \times \left[\sum_0^\infty (-)^n(2n+1)\mathrm{P}_n'(\mu)\frac{\mathrm{Q}_n(\nu_2)}{\mathrm{Q}_n'(\nu_1)}\right]^2 d\mu.$$

The evaluation of the pressure on the obstacle therefore depends on the integral

$$f(m,n) = \int_{-1}^{+1}\frac{\mu(1-\mu^2)}{\nu_1^2-\mu^2}\mathrm{P}_m'(\mu)\mathrm{P}_n'(\mu)\,d\mu.$$

This integral has recently been shown * to be equal to

$$-2(\nu_1^2-1)\mathrm{Q}_m'(\nu_1)\mathrm{P}_m'(\nu_1)$$

if $m-n$ is an odd integer, and zero if $m-n$ is an even integer. Thus we have

$$p = -\frac{4\pi\sigma m^2}{a^2}\mathrm{I},$$

where

$$\mathrm{I} = 3\frac{\mathrm{Q}_1(\nu_2)}{\mathrm{Q}_1'(\nu_1)}\mathrm{P}_1'(\nu_1)[5\mathrm{Q}_2(\nu_2)+9\mathrm{Q}_4(\nu_2)+\ldots] + 5\frac{\mathrm{Q}_2(\nu_2)}{\mathrm{Q}_2'(\nu_1)}\mathrm{P}_2'(\nu_1)[7\mathrm{Q}_3(\nu_2)+11\mathrm{Q}_5(\nu_2)+\ldots] + 7\frac{\mathrm{Q}_3(\nu_2)}{\mathrm{Q}_3'(\nu_1)}\mathrm{P}_3'(\nu_1)[9\mathrm{Q}_4(\nu_2)+13\mathrm{Q}_6(\nu_2)+\ldots].$$

This form for I admits simple approximations in terms of $1/\nu_2$. For

$$\mathrm{Q}_n(\nu_2)\mathrm{Q}_m(\nu_2) = \frac{n!\,n!}{(2n+1)!}\times\frac{m!\,m!}{(2m+1)!}\left(\frac{2}{\nu_2}\right)^{m+n+2}\left[\tfrac{1}{4}+\mathrm{O}\left(\frac{1}{\nu_2}\right)^2\right].$$

Thus the leading approximations will be

$$\mathrm{I} = \mathrm{O}\left(\frac{1}{\nu_2}\right)^5 = 15\frac{\mathrm{P}_1'(\nu_1)}{\mathrm{Q}_1'(\nu_1)}\mathrm{Q}_1(\nu_2)\mathrm{Q}_2(\nu_2)+\mathrm{O}\left(\frac{1}{\nu_2}\right)^7$$

* D. M. Wrinch, "Some Integrals involving Legendre Polynomials," Phil. Mag. x. (1930).

$$= 15\frac{P_1'(\nu_1)}{Q_1'(\nu_1)}Q_1(\nu_2)Q_3(\nu_2) + 27\frac{P_1'(\nu_1)}{Q_1'(\nu_1)}Q_1(\nu_2)Q_4(\nu_2) + 35\frac{P_2'(\nu_1)}{Q_2'(\nu_1)}Q_2(\nu_2)Q_3(\nu_2) + O\left(\frac{1}{\nu_2}\right)^9,$$

and so on. Alternatively, we may write them

$$I = 0,$$

neglecting $(1/\nu_2)^5$.

$$I = \frac{2}{3}\frac{P_1'(\nu_1)}{Q_1'(\nu_1)}\left(\frac{1}{\nu_2}\right)^5 = -\frac{2}{3}\frac{\nu_1^2-1}{\nu_2^5},$$

neglecting $(1/\nu_2)^7$.

$$I = \frac{2}{3}\frac{P_1'(\nu_1)}{Q_1'(\nu_1)}\left[1 + \frac{51}{35\nu_2^2}\right]\frac{1}{\nu_2^5} + \frac{8}{35}\frac{P_1'(\nu_1)}{Q_1'(\nu_1)}\cdot\frac{1}{\nu_2^7} + \frac{4}{15}\frac{P_2'(\nu_1)}{Q_2'(\nu_1)}\cdot\frac{1}{\nu_2^7}$$

$$= \frac{2}{3}\frac{P_1'(\nu_1)}{Q_1'(\nu_1)}\frac{1}{\nu_2^5} + \frac{6}{5}\frac{P_1'(\nu_1)}{Q_1'(\nu_1)}\frac{1}{\nu_2^7} + \frac{4}{15}\frac{P_2'(\nu_1)}{Q_2'(\nu_1)}\cdot\frac{1}{\nu^7},$$

neglecting $(1/\nu_2)^9$. Closer approximations can be obtained after the same manner.

XCII. *Notices respecting New Books.*

Conformal Representation. By C. Caratheodory. (Cambridge Tracts in Mathematics and Mathematical Physics.) (Cambridge University Press, 1932. Price 6*s.* 6*d.*)

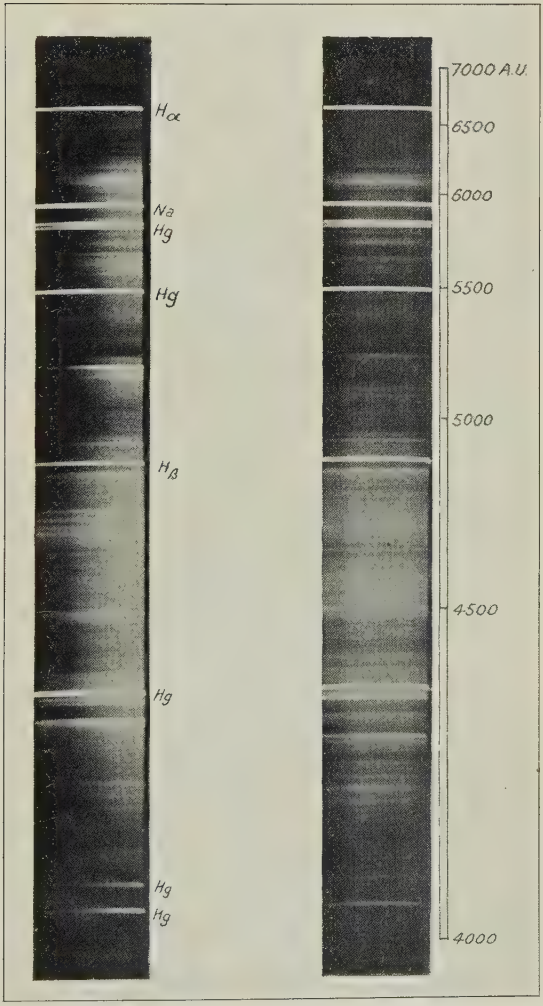
THIS tract will be useful to the student of pure mathematics and also to the large number of workers in mathematical physics, where conformal transformations play an important part.

The subject has of course a long and interesting history, certain conformal transformations being known (*e. g.*, the Euclidian magnification of the plane and the stereographic projection of the sphere) very early in history.

The aim of Professor Caratheodory has been to give a comprehensive account of the new methods elaborated in the last twenty years, which allow a simple treatment of the fundamental problems of the subject.

The tract is excellently translated from the German of the original manuscript by Dr. B. M. Wilson and Miss Margaret Kennedy.

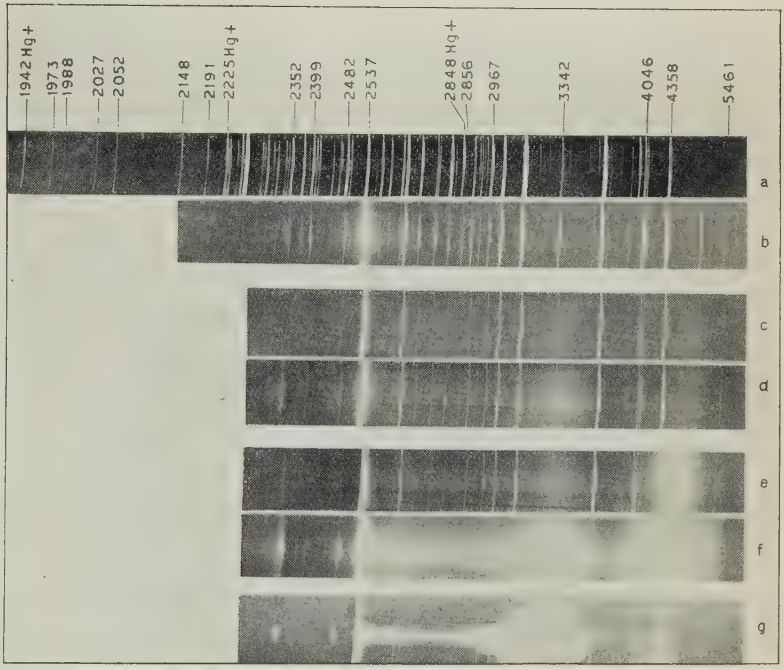
[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]



b. *a.*

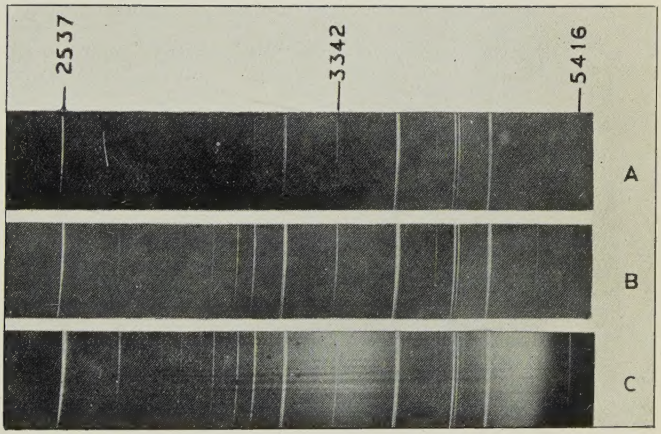
Spectrograms of tungsten cold cathode arc in high vacuum.

a. Pure tungsten.
b. Tungsten with borax.

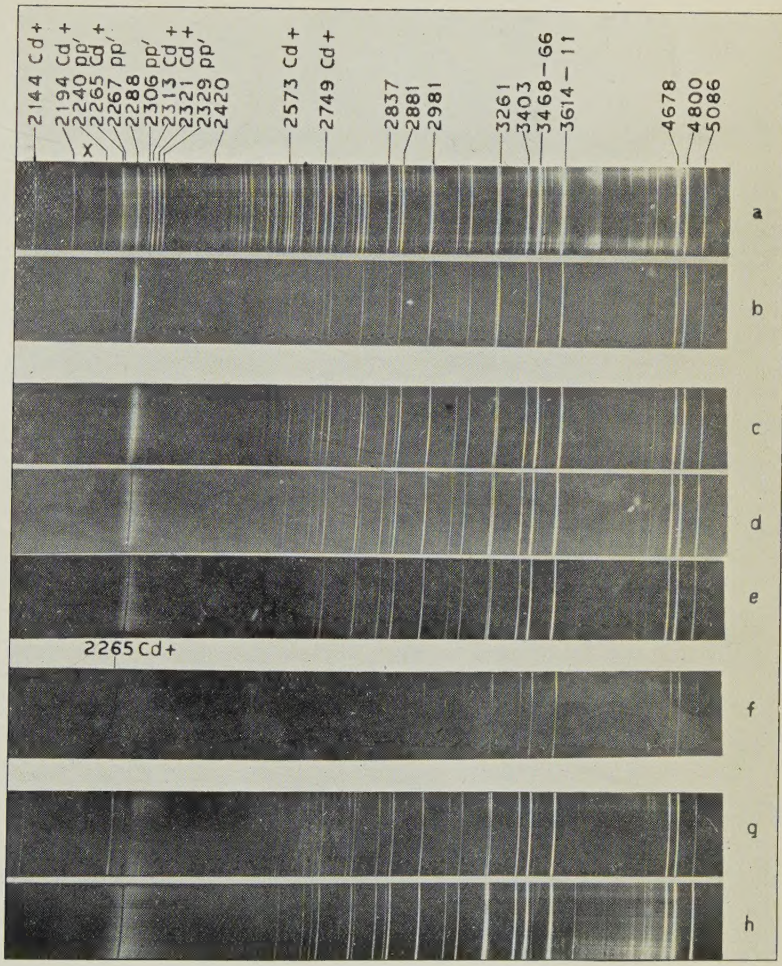


Mercury Spectrograms.

a., D. W. 87°– 95°, Exposure 3′.
b., C. W. 83°–104°, Exposure 20′.
c., D. W. 146°–152°, Exposure 5′.
d., C. W. 144°–147°, Exposure 5′.
e., D. W. 185°–195°, Exposure 8′.
f., C. W. 189°–195°, Exposure 5′.
g., C. W. 220°, Exposure 2′.



Mercury Spectrograms 1 mm. gap.
a., D. W. 90°, Exposure 1 hr.
b., D. W. 130°, Exposure 1 hr.
c., D. W. 160°, Exposure 1 hr.



Cadmium Spectrograms.

a., D. W. 300°–320°, Exposure 5′ intermittent.
b., C. W. 320°–370°, Exposure 15′.
c., C. W. 430°–440°, Exposure 15′.
d., C. W. 435°–470°, Exposure 25′.
e., C. W. 515°–535°, Exposure 20′.
f., D. W. 470°–507°, Exposure 30′ intermittent.
g., D. W. 550°, Exposure 6 hrs.
h., C. W. 550°, Exposure 6 hrs.